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ON THE SAMPLING ERRORS OF FACTOR LOADINGS

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The results of three empirical studies on the sampling fluctuation of centroid factor loadings are reported. The first study is based on data which happened to be available on 8 variables for 700 cases and which were factored to three factors for subsamples. The second study is based on fictitious data for 2500 cases which provided separate analyses on 25 samples for each of three situations: 5 variables, one factor; 5 variables, two factors; and 6 variables, three factors. The third study, based on real data for 9 variables and 7000 cases, involves separate factorization for 35 samples of 200 cases. The three studies agree in showing that the sampling behavior of first centroid factor loadings is much like that of correlation coefficients, whereas the sampling fluctuations for loadings beyond the first are disturbingly large.

Those who have been instrumental in the recent development of factorial methods to a high degree of mathematical nicety have been so casual in their treatment of sampling that many who are applying the new methods have failed to realize that the data for analysis are basically *statistical* in nature, and therefore subject to sampling errors. Sampling can affect the results of factor analysis *via* the stability of loadings, the number of factors to be extracted, and the location of the rotated factorial axes. This paper will be concerned primarily with the first of these, but the results will of course have implications for the third. The problem of sampling is that of determining not only the nature and extent of the distribution of successive sample values but also the presence of bias, i.e., whether the sampling distributions center about the universe or population parameters. It is not easy, however, to hypothecate a universe factor loading and conceive of sampling fluctuations therefrom, nor is it easy to imagine the sampling behavior of axes, either centroid or rotated.

Thurstone's assertion that simple structure will not arise as a result of chance sampling, though acceptable on theoretical grounds, is not sufficient in actual situations wherein the structure may be so lacking in conciseness as to involve a great deal of subjective arbitrariness. The struggle to put psychological meaning into the results of factor analyses has been greatly befuddled by the presence of chance

errors, the nature and magnitude of which are unknown. Perhaps many imaginative investigators have succeeded in assigning seemingly rational meaning to that which has resulted largely from chance, and, therefore, some of the traits supposedly isolated may be only imaginative. Aside from the need to know something of the magnitude of sampling errors in studies directed primarily at the isolation of traits or abilities, there is even greater need for this information in connection with such problems as the relationship of organization to age, the effect of changes experimentally produced (e.g., practice), and the invariability of the factorial description of tests which are moved from one battery to another when the two batteries have been administered to two different groups of subjects.

Ideally the size of sampling errors in factorial analysis should be determined analytically, but the non-analytic nature of certain steps in the Thurstone method, the method which at present seems most defensible psychologically, precludes the analytic approach. It would seem appropriate to attack the problem empirically. The writer's first approach was by way of data which just happened to be available. A group of 700 cases with scores on 8 variables was broken down into 14 subsamples of 50 each and 7 subsamples of 100 each. The product moment correlational matrices were factorized to three factors. The results can be summarized briefly by saying that the first centroid loadings tended to fluctuate to about the same extent as product moment correlation coefficients and that the second and third centroid loadings showed greater fluctuation — about twice that of the first factor loadings.

Fictitious Data

Our second approach consisted in setting up "scores" on 7 variables for 2500 cases in such a manner that 5 variables with one factor, 5 variables with two factors, and 6 variables with 3 factors could be chosen for analysis. The scores were obtained by counting the number of odd digits in columns of Barlow's Tables (Tippett's random numbers are not extensive enough for our purpose), one column corresponding to an individual, and 15 columns being utilized per page beginning with page 22. The variables were defined in terms of rows, and the numbers of odds were counted and summed so as to provide common elements between some variables and elements specific to each variable. The set-up in terms of elements is presented in Table 1.

It was assumed that a digit's being odd (or even) is a chance matter, but of course this isn't true — some columns were not used as cases because of obvious runs. Now this lack of randomness can be

expected to operate so as to interfere with the obtaining of 2500 cases with scores, and consequent intercorrelations, of exactly the factorial composition indicated in Table 1. This would invalidate these data for use in setting up a criterion for the number of factors, but so far as the present sampling problem is concerned, we need not worry if the 2500 cases do not yield intercorrelations sufficiently in accord with the theoretically expected correlations. As a matter of fact, some of the empirical r's based on all 2500 cases are significantly different from the expected values. These discrepancies, indicating definitely that the numbers in Barlow do not exhibit randomness as regards being odd or even, do not nullify the data for our purpose. In other words, when we have determined scores for 2500 cases on the 7 variables, we can safely ignore this inadequacy in setting up the scores, providing our method for breaking the 2500 cases down into subsamples is random and independent of the scheme utilized in setting up the scores.

The 2500 cases were randomly (we believe) divided into 25 groups of 100 each by the following procedure. The cases were serially numbered from 1 to 2500, sorted into 10 piles of 250 each on the basis of the digit's column of the serial number on the Hollerith cards, then each pile of 250 was sorted on the basis of the ten's column into 10 groups of 25 each, and finally four such small groups were combined for samples of 100 each. The writer sees no reason why this should not lead to random samples.

The intercorrelations for the successive 25 samples and for the total group were computed as tetrachoric r's with cuts near the medians. The use of tetrachorics will of course lead to larger sampling errors than would obtain if product moment r's were involved. The 7 variables had been arranged so as to permit the following series of factor analyses: five variables (1, 2, 3, 4, and 7) with one factor; five variables (1, 2, 4, 5, and 6) with two factors; six variables (1, 3, 4, 5, 6, and 7) with three factors. For each of these series, 26 analyses were carried out, thus permitting the 25 sets of subsample loadings to be compared with the loadings obtained by analyzing the total group. No corrections for communality estimates were utilized. (For one series, the several analyses were carried through on the basis of second or revised estimates, but this did not lead to an appreciable change in the results.)

A word should be inserted about sign changing or test reflection. The procedure followed was a combination of the number of negative signs and the ultimate criterion that column sums should not be negative. As regards the proper sign to be given to the loadings, an attempt was made to have the signs in each column in the sample

factorial matrices be in maximum agreement with the signs in the corresponding matrices based on the total 2500.

The results of the several series are reported in Tables 2, 3, and 4. The entries in these tables are nearly self-explanatory: the rows marked 2500 give the loadings which resulted from factorizing the correlations based on the total group; the next row gives the mean loading for the 25 samples; and S.D. is the sigma of the sampling distribution of loadings. These S.D.'s presumably approximate the standard errors of the centroid factor loadings.

It will be noted from these tables that the first factor loadings exhibit sampling fluctuations of about the same order as the chance errors of tetrachoric correlations of magnitudes corresponding to the factor loadings and based on N's of 100. It will also be noted that there is some tendency for the size of these sampling errors to be dependent upon the magnitude of the "universe" loading. This parallels the known sampling behavior of correlation coefficients. The second and third factor loadings show rather marked fluctuations. If these data are at all valid, it would seem that loadings beyond the first are rather unstable from the sampling viewpoint.

The results in Tables 2, 3, and 4 are in terms of centroid loadings. One wonders what would happen in case the axes were rotated, but certain difficulties arise here—the sampling fluctuations of projections on rotated axes would involve taking deviations from axes which would themselves be subject to sampling variability plus arbitrariness in the rotations. It is here that the fairly close accord between the means of 25 sample loadings and those of the "universe" loadings, based on all 2500 cases, takes on importance. It would seem that the centroid projections do fluctuate about the "universe" centroid loadings.

Aside from the sampling distribution of loadings, one further question may be raised. What is the consequent effect of sampling on the test clusters? Or to what extent do the tests tend to show the same clustering for a sample as for the "universe" from which the sample is drawn? With regard to the five variable, two factor series, it will be noted that the set-up for these variables (see Table 1) is such that, theoretically, one would expect an axis through tests 1 and 2 to be orthogonal to an axis through tests 5 and 6, and that test 4 would fall between these axes. This holds true for the analysis based upon all 2500 cases, but when the factor plots for the several 25 samples were examined it was discovered that five samples yielded clusters decidedly different from the "universe" pattern. The remaining 20 samples show fair agreement, i.e., one would arrive at an interpretation therefrom which would not be much at variance with that deduced from the "universe."

When a similar examination was made of the factor patterns for the six variable, three factor set-up, it was found that in at least 10 of the 25 samples there would be questions as to whether the clusters for the samples agreed with that for the "universe." These divergences, it should be noted, cannot be rectified by rotations. Further discussion will be postponed until after the results of a third approach are presented.

*College Entrance Board Data**

The scores for 7000 cases on 9 variables were punched on Hollerith cards in such a manner as to permit the determination of tetrachorics with near median cuts. The case numbers from the test booklets were also included on the cards, which were serialized by these numbers, and then sorted by hand into 35 piles of 200 cards each, the 1st card to pile 1, the 2nd to pile 2, $\cdots$, the 35th to pile 35, the 36th to pile 1, etc. The intercorrelations for these 35 random samples and for the total 7000 were computed, and the 36 resulting correlational matrices were factorized to three factors. Although the magnitudes of the residuals indicate rather definitely that at least three factors have statistical significance for the "universe," i.e., the total 7000, one may well question the significance of more than two factors for the samples of size 200. It is of interest, however, to know how widely the third factor loadings vary from sample to sample even though a possibly valid criterion for the number of factors would suggest that a third factor should not be extracted for these samples of 200 cases.

It should again be noted that the signs in the columns of each of the several factorial matrices were adjusted by changing all the signs, if necessary, in a column so as to get the maximum sign agreement with the corresponding column of the "universe" factorial matrix.

The distributions for the factor loadings resulting from the 35 successive samples will be found in Tables 5, 6, and 7. At the bottom of each distribution will be found the "universe" loadings, the means for the sampling values, and the sigmas of the distributions. (The means and sigmas were computed from distributions with finer groupings). From these tables, there would seem to emerge a number of pertinent facts, facts which are consistent with the published and unpublished results of the less adequate fictitious data.

In the first place, the means for the first centroid loadings agree quite closely with the "universe" loadings, the largest discrepancy being .007, whereas the correspondence of the means for second and third loadings is not particularly good.

* The writer is grateful to Professors John M. Stalnaker and Carl C. Brigham for these data.

In the second place, the sampling distributions for the first factor loadings show dispersions which are comparable with that expected for the sampling distribution of correlation coefficients (tetrachoric) based on samples of, in this case, 200. As regards loadings for factors beyond the first, it will be noted that the dispersions for the second centroid loadings are more than double those for the first, while the third factor projections yield distributions which have standard deviations approximately three times as large as those for the first axis.

In the third place, there are certain features of the distributions which need to be mentioned. There is a noticeable tendency for the distributions in Table 5 to be skewed in much the same fashion as might be anticipated if the sampling behavior of high factor loadings were similar to that for high correlation coefficients. Nearly all the distributions for the second and third factors exhibit a marked tendency towards bimodality. It follows, therefore, that means and sigmas are not sufficient as descriptive measures for these distributions. The writer, without applying significance tests, believes that the apparent bimodality is real, that it is inherent in the centroid method, and that it is largely due to the necessity for reflecting tests so as to remove the centroid from zero for residual tables. It should be obvious that, for any one test, the signs for all 35 sample factor loadings could be made to agree by an appropriate arrangement of all the signs in the proper column of the several sample factorial matrices and thus the bimodality for that test could be eliminated. Such a procedure, however, would not lead to as close agreement with the "universe" values for all tests considered simultaneously. As already indicated, the distributions in Tables 6 and 7 represent the maximum amount of agreement between sample and universe values as regards the signs of the loadings. Any one test could be brought into closer agreement but only at the expense of poorer agreement for some other test or tests.

Aside from these features of the sampling distributions, let us consider the effect of sampling on the test clusters or factor patterns. An examination of the plots reveals that no less than 15 of the 35 samples have yielded clusters which are different from that for the universe. This is somewhat worse than that found for the fictitious data despite the fact that for the College Entrance Board Data we have a larger number of variables and larger samples. This may be due to the fact that the fictitious data were so set up that definite significance could be attached to the second and third factors, whereas the significance or meaning of factors beyond the first for the Board Data is open to question. For the total 7000 cases, tests 2 and 6 (both numerical) tend to hang together, but aside from this the writer has been

unable to make sense of the factor pattern. Perhaps, therefore, in examining the patterns for the 35 successive samples, one should pay particular attention to the behavior of tests 2 and 6. When this is done, it is found that these two tests do tend to hang together from sample to sample, but one wonders what significance can be assigned to this fact in those sample situations wherein other tests irrationally join the numerical "cluster."

It might be claimed that no significance can be assigned to the second and third factor loadings for the universe from which the samples were drawn, and consequently that the results for the samples are trifling. This would be equivalent to saying that centroid loadings as high as .40 to .60 are trifling even though based on a sample of 200 (equivalent in sampling effectiveness to about 100 cases when product moment r's are used). The important point, in the writer's opinion, is that the sampling fluctuation of loadings for factors beyond the first is apparently so great that large loadings may readily occur by chance. Incidentally, the writer is not convinced that the results of analyses based on relatively small samples have more than chance significance merely because an investigator, after an obvious struggle, succeeds in assigning supposedly rational meanings to his findings.

Discussion

The results of these studies on the effect of sampling on factor loadings are consistent in showing that first factor loadings behave like correlation coefficients, whereas loadings beyond the first not only show greater fluctuation but also tend to bimodality. The implications of these findings, taken at their face value, would seem quite obvious. There are, however, certain limitations to these studies which need to be stressed, the most serious of which is the smallness in the number of variables. The inclusion of more variables will not necessarily iron out the effect of the chance errors of the original correlation coefficients since these sampling errors are not independent, i.e., they are correlated. One might expect that the use of additional or revised estimates of communalities would result in slightly greater stability for the factor loadings. The results herein are limited to Thurstone's centroid method. Whether other factorial methods are subject to similar sampling fluctuations remains to be determined.

There is one point of theoretical interest which rapidly goes beyond the mathematical comprehension of the present writer. In other sampling situations it is usually easy to conceive of a universe value about which sample values will disperse themselves. In the factorial

situation, it is difficult to see just what the universe value or values might be. The statistic which varies from sample to sample is the factor loading or projection on a centroid axis which itself varies from sample to sample. A mere rotation of axes, regardless of how clear-cut the structure, will not change the predicament—the location of the rotated axes is subject to the chance vagaries of sampling, and of course, the test projections thereupon are also affected by sampling. In other words, we have no stable reference frame from which to regard the sampling fluctuations of test projections. Suppose we have n variables which for the universe yield a simple structure of exactly r primaries. A sample of 200 is drawn randomly. Neither the centroid nor the rotated axes can be thought of as necessarily coinciding with the axes for the universe, and therefore the deviations of the loadings from universe values will not be determinate. This may constitute a serious theoretical limitation to our empirical study.

If the results of these empirical studies are to be taken as even approximating the truth, it follows that psychologists might very well cease filling the literature with factor analyses based on, say, less than 100 cases. Furthermore, if the sampling situation in factor analysis is anything like that in other branches of statistics, one might very well expect that somewhere along the line the concept of degrees of freedom will enter in such a manner that the effective size of the sample is diminished by the number of variables in the analysis. The writer is not anxious specifically to cite any particular study, but he does find himself wondering about the sampling effectiveness of a factor study of 52 variables based on tetrachoric r's for 110 cases. The use of tetrachorics automatically reduces the effective size of the sample to about 55, whence there may be left but 3 degrees of freedom or an effective N of only 3. Perhaps this explains why so much difficulty was encountered when trying to make psychological sense of the results.

TABLE 1

Number of common and specific elements for fictitious data

Variable	A	B	C	S_i	T_i
1	5			2	7
2	5			8	13
3	5			10	15
4	5	2		2	9
5		2		5	7
6		2	3	4	9
7	5		3	3	11

TABLE 2

Five variables, one factor

First factor

	1	2	3	4	7
2500	.83	.69	.69	.81	.76
Mean	.83	.69	.70	.80	.75
S.D.	.07	.12	.10	.09	.11

TABLE 3

Five variables, two factors

First factor

	1	2	4	5	6
2500	.73	.63	.86	.25	.32
Mean	.71	.63	.85	.29	.34
S.D.	.14	.14	.09	.18	.25

Second factor

	1	2	4	5	6
2500	.42	.33	.06	—.36	—.37
Mean	.41	.36	.09	—.45	—.36
S.D.	.16	.17	.19	.13	.21

TABLE 4

Six variables, three factors

First factor

	1	3	4	5	6	7
2500	.76	.61	.86	.18	.33	.76
Mean	.75	.61	.85	.22	.34	.74
S.D	.12	.11	.09	.14	.23	.12

Second factor

	1	3	4	5	6	7
2500	.33	.36	—.09	—.44	—.35	.22
Mean	.37	.36	.00	—.48	—.40	.20
S.D.	.15	.20	.20	.15	.19	.20

Third factor

	1	3	4	5	6	7
2500	.18	.10	.20	.14	—.31	—.28
Mean	.10	.02	.10	.28	—.26	—.16
S.D.	.24	.27	.23	.15	.12	.26

TABLE 5

College Entrance Board Data
Sampling distribution for first factor loadings

	1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
.88									1
.86									2
.84					6			4	
.82	5				5		3	2	6
.80	6		3		7		3	4	3
.78	5		2		4		4	3	5
.76	5		3		4		8	6	3
.74	1		5	2	6		3	5	8
.72	4		7		1		6	2	4
.70	3		6	1	1		2	4	2
.68	3		2	1			1	2	
.66	2		2	2	1			3	1
.64	1	4	1	3			3		
.62		4	1	4		4	1		
.60		5	2	2		2	1		
.58		4	1	7		3			
.56		3		5		2			
.54		2		1		6			
.52		5		1		7			
.50		1		4		2			
.48		2		1		5			
.46		3		1					
.44		1				2			
.42		1				2			
7000	.762	.562	.716	.592	.795	.540	.739	.763	.783
Mean	.763	.564	.720	.601	.792	.540	.746	.763	.781
S.D.	.053	.062	.056	.070	.045	.055	.058	.054	.049

TABLE 6

College Entrance Board Data
Sampling distribution for second factor loadings

	1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
.45					3		1		
.40			1	1	9				2
.35				1	8				5
.30			1	6	3		3	1	3
.25	1		5	3	1		2	1	2
.20	4		3	5	5		6	1	9
.15	2		6	3	4		6	7	4
.10	4		5	5			5	2	3
.05	6		3	2			3	6	2
.00	1		1	1			1	2	
—.05							1	1	
—.10	3	1	3		1		4	5	
—.15	3	1	2	3	1		2	2	4
—.20	3		1	3			1	3	
—.25	6	3	3			2		2	
—.30	2	2	1	1		2		2	
—.35		4		1		5			1
—.40		4				3			
—.45		3				7			
—.50		5				6			
—.55		8				5			
—.60		2				4			
—.65		2				1			
7000	.059	—.474	.180	.160	.325	—.451	.157	—.063	.141
Mean	—.016	—.414	.087	.149	.315	—.434	.129	.021	.193
S.D.	.173	.134	.178	.181	.135	.101	.148	.162	.176

TABLE 7

College Entrance Board Data
Sampling distribution for third factor loadings

	1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
.45				1			1		
.40			1	3					
.35				3		2	2		
.30	2	2		3		7	1		
.25	3	1	6	6	2	4	5	3	
.20	3	2	3	6	1	1	6	3	
.15	3	4	2	2		6	4	3	
.10	5	1	6	3	4	5	3	2	
.05	5	3	1	2	1		4	2	
.00			1				1		
—.05								1	
—.10	1	5	1		2	2	4	5	2
—.15	3	2	2	2	4	3	1	1	5
—.20	1	7	2	2	3	2	1	9	3
— 25	3	1	3		8	1	1	2	10
—.30	2	4	5	1	6	1	1	3	3
—.35	2	2	1		3	1			3
—.40	2		1	1	1			1	6
—.45									2
—.50		1							1
7000	.157	.067	.179	.134	—.237	.204	.125	—.162	—.368
Mean	.018	—.051	.021	.186	—.135	.129	.132	—.036	—.252
S.D.	.219	.210	.228	.212	.177	.201	.180	.190	.105

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TEST RELIABILITY ESTIMATED BY ANALYSIS OF VARIANCE

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A formula for estimating the reliability of a test, based on the analysis of variance theory, is developed and illustrated. The data needed for the required computation are the number of correct responses to each item and the score for each subject. The results obtained from this formula are identical with those from one of the special cases of the Kuder-Richardson formulation. The relationships of the new procedure to other approaches to the problem are indicated.

This paper is composed of two parts, the first of which describes the procedures followed in computing the estimated reliability of a test by analysis of variance while the second part contains the mathematical derivation of the formulas.

I

The coefficient of reliability of a test gives the percentage of the obtained variance in the distribution of test scores that may be regarded as true variance, that is, as variance not due to the unreliability of the measuring instrument. Out of Rulon's *work on a short method of estimating the reliability coefficient by means of "split-halves" comes the relationship

$$r_{tt} = 1 - \frac{\sigma_d^2}{\sigma_t^2}$$

where σ_d^2 is the variance of the distribution of differences obtained by subtracting a student's score on the odd items of a test from that on the even items, or vice versa, and σ_t^2 is the variance of the distribution of the students' scores on the test. From this information, it is apparent that σ_d^2 is used as a measure of the discrepancy between the obtained variance and the true variance, σ_∞^2. If the odd-even split of the test happens to be an unlucky one, σ_d^2 as thus computed may be an underestimate or an overestimate of the discrepancy between the obtained variance and the true variance. It was this difficulty that led

* Rulon, Phillip J. A simplified procedure for determining the reliability of a test by split-halves. *Harvard Educational Review*, 1939, 9, 99-103.

the writer to seek a better estimate of this discrepancy. Table I shows the results of a hypothetical test where each item is scored either one or zero.

TABLE I

	Items				
Student	1	2 ·	3	n	Scores
1					t_1
2					t_2
3					t_3
.					.
.					.
.					.
k					t_k
Totals	p_1	p_2	p_3	p_n	$\sum_{i=1}^{n} p_i = \sum_{i=1}^{k} t_i$

The sum of squares "among students" is

$$\frac{1}{n}\sum_{i=1}^{k} t_i^2 - \frac{\left(\sum_{i=1}^{k} t_i\right)^2}{nk} . \tag{1}$$

The sum of squares "among items" is

$$\frac{1}{k}\sum_{i=1}^{n} p_i^2 - \frac{\left(\sum_{i=1} t_i\right)^2}{nk} . \tag{2}$$

The total sum of squares is

$$\frac{\left(\sum_{i=1}^{k} t_i\right)\left(nk - \sum_{i=1}^{k} t_i\right)}{nk} , \tag{3}$$

or simply the number of correct responses times the number of incorrect responses divided by the total number of responses, nk. This last result can be verified as follows.

Let n_1 equal the number of correct responses and n_2 equal the number of incorrect responses. Then the mean is $\dfrac{n_1(1) + n_2(0)}{n_1 + n_2}$ or $\dfrac{n_1}{n_1 + n_2}$. The sum of the squares of the deviations from the mean is

$$n_1\left(1 - \frac{n_1}{n_1 + n_2}\right)^2 + n_2\left(\frac{n_1}{n_1 + n_2} - 0\right)^2 = \frac{n_1 n_2^2 + n_1^2 n_2}{(n_1 + n_2)^2} \text{ or } \frac{n_1 n_2}{n_1 + n_2} .$$

In terms of the notation of the analysis of variance problem above, this sum is clearly (3).

By subtracting the "among students" and the "among items" sums of squares from the total sum of squares, we have left the residual sum of squares which is used as the basis of estimating the discrepancy between the obtained variance and the true variance. This estimate of the discrepancy is a better one than that obtained by dividing the test into odd and even halves because in the latter case the particular split of the test, which is only one of many possible ways of splitting a test, may be an unlucky division and may result in either an overestimate or an underestimate of the coefficient of reliability. Furthermore, it has been shown, as the reader will see in Part II, that the particular estimate of the discrepancy between the obtained and the "true" scores is the best linear estimate where "best" is considered in the light of the least squares' criterion. Hence, it is clear that this method of estimating the reliability of a test gives a better estimate than any method based upon an arbitrary division of the test into halves or into any other fractional parts. From a practical point of view, the labor involved in computation is not excessive although probably not much less than is required for the computation by split-halves. The data needed to compute the reliability are the number of correct responses to each item and the score for each subject. The use of the item counter on the International Scoring Machine makes it possible to obtain these item counts immediately.

The method of obtaining the coefficient of reliability will be illustrated by a specific example. A test of 250 items was administered to 33 students in a class in botany in the College of Pharmacy of the University of Minnesota.

TABLE II

Analysis of Variance of Test of 250 Items in Botany, Administered to 33 Students

Source of Variation	d.f.	Sum of Squares	Variance
Total	8249	2002.42533	.24275
Among Items	249	593.81927	2.38482
Among Individuals	32	82.82933	2.58842
Remainder	7968	1325.77673	.16639

The coefficient of reliability of the test may be expressed

$$r_{tt} = \frac{2.58842 - .16639}{2.58842} = .936\ .$$

The standard error of measurement is $\sqrt{1325.77673/32}$ or 6.44.

That measure of reliability which Jackson* has called the sensitivity, γ, can be obtained immediately as,

$$\gamma^2 = \frac{2.42203}{.16639} = 14.556 \text{ and } \gamma = 3.82 .$$

γ is interpreted on the normal probability scale as follows. If $\gamma = 2.57$ for example, using the obtained scores as estimates of a certain capacity of the individual, we would expect to make an error as great as or greater than one standard deviation (of the true scores) only once in a hundred times.

It may be interesting to some who are familiar with the work of Kuder and Richardson† that the foregoing method of estimating the coefficient of reliability gives precisely the same result as formula (20) of their paper. This fact can be easily verified algebraically.

If two or more equivalent forms of the test are administered to the same group of students it is easy to extend this procedure so that it is possible to separate out another source of variation, the "between forms" variance which might be considered to be due to practice effect.

More extended examination of the "among items" variance would make it possible to decide on heterogeneity of the respective difficulties of the items while a more extended examination of the "among students" variance would make it possible to answer certain pertinent questions regarding the individual differences among students.

II

The method used in developing the formulas for estimating the reliability of a test by means of analysis of variance as described in Part I is essentially the method suggested by Johnson and Neyman‡ and later used by Jackson§. This particular approach does not give new or different results for the problems of tests of significance but does possess considerable advantage in attacking problems of estimation. The problem of chief concern here is clearly a problem of estimation, since only rarely would the reliability of a test be so low that its difference from zero would be a major issue, although this could be tested by the use of the variance ratio $[(n-1)S_2 - S_0]/S_0$. (See

* Jackson, Robert W. Reliability of mental tests. *British Journal of Psychology*, 1939, 29, 267-87.

†Kuder, G. F. and Richardson, M. W. The theory of the estimation of test reliability. *Psychometrika*, 1937, 2, 151-60.

‡ Johnson, Palmer O. and Neyman J. Tests of certain linear hypotheses and their application to some educational problems. *Statistical Research Memoirs*, 1936, 1, 57–93.

§ Jackson, *op. cit.*

equations (9) and (17) for definitions of these quantities).

Assume that the score, X_{is}, of the s-th student on the i-th item of the test may be represented as the sum of four independent factors or components (i.e., the vector X_{is} is resolved into four mutually perpendicular components). These four components may be described as follows:

(1) a component common to all individuals and to all items;
(2) a component associated with the item;
(3) a component associated with the individual;
(4) an error component that is independent of (1), (2), and (3) but includes a multitude of small variations produced by a multitude of causes which, though each by itself is relatively unimportant and unpredictable, taken together may be thought of as being distributed normally with variance σ^2, the precise value of which is unknown.

This may be expressed in mathematical form by assuming that

$$X_{is} = A + t_i + p_s + y_{is},$$

where $i = 1, 2, \cdots, n$ and $s = 1, 2, \cdots, k$, represents the score of the s-th individual on the i-th item of the test. Here A is the component common to all individuals irrespective of the particular item under consideration; it will be shown to be a constant for all students and all items. In the foregoing expression for X_{is}, t_i is the component associated with the item and p_s is the component associated with the student. (It will be seen later that $(A + t_i)$ is the quantity often called the difficulty of the item.) If a student were an "average student" his "true score" on the i-th item would be $(A + t_i)$; however, if he were above "average" his "true score" would be higher than $(A + t_i)$, such as $(A + t_i + p_s)$. (If he were below "average" p_s would be negative.)

We must further assume that the error component, y_{is}, of the i-th item is normally distributed with the same variance, σ^2, as is the error component of every other item, and that y_{is} is independent of y_{js} where $i \neq j$.

Then, since

$$y_{is} = (X_{is} - A - t_i - p_s) \qquad (1)$$

is distributed normally with standard deviation, σ,

$$\sum_{i=1}^{n} \sum_{s=1}^{k} y^2_{is} = \sum_{i=1}^{n} \sum_{s=1}^{k} (X_{is} - A - t_i - p_s)^2$$

is distributed as χ^2.

Hence

$$\chi^2 = \sum_{i=1}^{n} \sum_{s=1}^{k} (X_{is} - A - t_i - p_s)^2 \tag{2}$$

The reliability coefficient of a test is the ratio of the variance of the "true scores" to the variance of the obtained scores, or in other words, gives the percentage of the obtained variance that may be spoken of as "true" variance or not due to the unreliability of the test. If we let σ^2_t represent the variance of the obtained scores and σ_d^2 the discrepancy between the variance of the obtained scores and the variance of the "true" scores, the reliability coefficient is given by the ratio: $r_{tt} = (\sigma^2_t - \sigma^2_d)/\sigma^2_t$. The variance of the error term, y_{is}, or σ^2_d, is the expression of which we wish to obtain the best estimate. According to Markoff's theorem,* the best linear estimate of y_{is} can be obtained, in the situation where the σ's of the independent observations are all equal, by minimizing the sum of squares (2) with respect to A, t_i, and p_s as independent variables and substituting these values, A°, t_i°, and p_s°, in (1) to give y_{is}', the best linear estimate of the discrepancy between the obtained score and the "true" score.

A first necessary condition for minimizing (2) is that the partial derivatives with respect to A, t_i and p_s must vanish.

$$\frac{\partial \chi^2}{\partial A} = -2 \sum_{i=1}^{n} \sum_{s=1}^{k} (X_{is} - A - t_i - p_s); \tag{3}$$

$$\frac{\partial \chi^2}{\partial t_i} = -2 \sum_{s=1}^{k} (X_{is} - A - t_i - p_s) \quad \text{for each } i = 1, 2, \cdots, n; \tag{4}$$

$$\frac{\partial \chi^2}{\partial p_s} = -2 \sum_{i=1}^{n} (X_{is} - A - t_i - p_s) \quad \text{for each } s = 1, 2, \cdots, k. \tag{5}$$

Setting each of these partial derivatives equal to zero and solving simultaneously gives the values for A, t_i, and p_s which minimize χ^2. These values of A, t_i, and p_s which render χ^2 a minimum will be designated by A^0, t_i^0, and p_s^0.

$$A^0 = \frac{1}{nk} \sum_{i=1}^{n} \sum_{s=1}^{k} X_{is} - \frac{1}{n} \sum_{i=1}^{n} t_i^0 - \frac{1}{k} \sum_{s=1}^{k} p_s^0; \tag{6}$$

$$t_i^0 = \frac{1}{k} \sum_{s=1}^{k} (X_{is} - p_s^0) - A^0 \quad (i = 1, 2, \cdots, n); \tag{7}$$

* Neyman, J. The Markoff Method and Markoff Theorem on Least Squares. *Journal of the Royal Statistical Society*, 1934, 97, 593-594.

$$p_s^0 = \frac{1}{n}\sum_{i=1}^{n}(X_{is} - t_i^0) - A^0 \quad (s = 1, 2, \cdots, k) \tag{8}$$

Substituting these values of A^0, t_i^0, and p_s^0 in (2) gives the minimum value of χ^2 which is designated by S_0.

$$S_0 = \sum_{i=1}^{n}\sum_{s=1}^{k}(X_{is} - A^0 - t_i^0 - p_s^0)^2$$

$$= \sum_{i=1}^{n}\sum_{s=1}^{k}(X_{is} - \bar{x}_{i.} - \bar{x}_{.s} + \bar{x})^2 , \tag{9}$$

where

$$\bar{x}_{i.} = \frac{1}{k}\sum_{s=1}^{k}X_{is} \; ; \bar{x}_{.s} = \frac{1}{n}\sum_{i=1}^{n}X_{is} \; ; \quad \bar{x} = \frac{1}{nk}\sum_{i=1}^{n}\sum_{s=1}^{k}X_{is} . \tag{10}$$

Substituting these values of A^0, t_i^0, and p_s^0 in (1) gives the best linear estimate of y_{is} (i.e., the error component of the response of the s-th student to the i-th item).

Thus

$$y'_{is} = X_{is} - \bar{x}_{i.} - \bar{x}_{.s} + \bar{x} , \tag{11}$$

Since $\sum_{i=1}^{n}\sum_{s=1}^{k} y'_{is} = 0$, $S_0 = \sum_{i=1}^{n}\sum_{s=1}^{k}(y'_{is} - \bar{y})^2$, so that S_0 is f times the variance of the y'_{is}, the error component, where f is the number of degrees of freedom or the number of independent variates necessary to express the sum, S_0. It is clear that $f = (n-1)(k-1)$ if we consider the nk X_{is}'s arranged in n rows and k columns, so there are $(n-1)$ independent variates in each of the $(k-1)$ columns. Hence the best estimate of the variance of the error component is

$$\frac{S_0}{(n-1)(k-1)} . \tag{12}$$

In order to determine the best estimate of σ^2_s, the variance of the component p_s associated with the student, assume p_s to be normally distributed with variance σ^2_s. Since p_s is independent of y_{is}, $\sum_{i=1}^{n}\sum_{s=1}^{k}(y_{is} + p_s)^2$ is distributed as χ^2.

Then

$$\chi^2 = \sum_{i=1}^{n}\sum_{s=1}^{k}(X_{is} - A - t_i)^2; \tag{13}$$

$$\frac{\partial \chi^2}{\partial A} = -2\sum_{i=1}^{n}\sum_{s=1}^{k}(X_{is} - A - t_i) ; \tag{14}$$

$$\frac{\partial \chi^2}{\partial t_i} = -2\sum_{s=1}^{k}(X_{is} - A - t_i) \qquad (i=1,2,\cdots,n.) \tag{15}$$

Setting each of these partial derivatives equal to zero and solving simultaneously gives the values of A' and t'_i which, when substituted in (13), will give $\sum_{i=1}^{n}\sum_{s=1}^{k}(y_{is} + p_s)^2$ its minimum value which is designated here as S_1.

Then

$$S_1 = \sum_{i=1}^{n}\sum_{s=1}^{k}(X_{is} - \bar{x}_{i.})^2. \tag{16}$$

This sum of squares gives the basis for estimating the variance σ_s^2, since $(S_1 - S_0)$ is the "among students" sum of squares.

Let

$$S_2 = S_1 - S_0. \tag{17}$$

Then $\frac{S_2}{k-1}$ is the obtained variance "among students." The obtained variance "among students" is not the best estimate of the true variance in case the hypothesis of homogeneity of the sample has been refuted as will usually be the case in test item responses. Instead, according to the work of Irwin,* the best estimate of the variance "among students" is obtained by subtracting the estimate of the error variance $\frac{S_0}{(n-1)(k-1)}$ from the obtained variance "among students."

Thus

$$\frac{S_2}{k-1} - \frac{S_0}{(k-1)(n-1)} \tag{18}$$

is the best estimate of the true variance of the students' responses. Hence the percentage of the obtained variance that is not associated with unreliability of the test is

$$r_{tt} = \frac{\dfrac{S_2}{(k-1)} - \dfrac{S_0}{(k-1)(n-1)}}{\dfrac{S_2}{k-1}} \tag{19}$$

or

$$\frac{(n-1)\,S_2 - S_0}{(n-1)S_2}. \tag{20}$$

Formula (19) is the one used for the computation of estimated test reliability in Part I.

* Irwin, J. O. Mathematical theorems involved in analysis of variance. *Journal of the Royal Statistical Society*, 1931, 94, 284-300.

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A FACTORIAL STUDY OF NUMBER ABILITY

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In order to investigate certain hypotheses concerning the nature of number ability, and, secondarily, the nature of perceptual speed, a battery of thirty-four tests was given to 223 Chicago high school seniors and the data were factored by the centroid method. Seven primary factors were identifiable upon rotation. Several deductions are made relative to the interpretation of the factors and relative to the consistency of the data with the hypotheses which were to be tested.

I. Introduction

Since the development of multiple factor methods by L. L. Thurstone, at least nine multiple-factor studies of cognitive abilities have been carried out in his laboratory. The general results of these studies have been in very high agreement. The same factors have occurred repeatedly in different test batteries with subjects from eighth-grade level to college freshmen. One of the clearest of the primaries and one that has been repeatedly obtained is number ability, characteristically defined by tests of addition and multiplication. These two tests are referred to as "simple" tests in that they have a large projection on only one or only a very few common factors.

It is not very satisfying to define an ability in terms of the content of the tests which measure that ability. This procedure should be regarded as a makeshift until investigations may be carried out which give insight into the psychological nature of the fundamental process characterizing the ability. For example we may ask: Is the ability defined by addition and multiplication specific to numerical manipulations or does it represent some more fundamental unity that transcends numerical operations? Is it inherent in the organism or a product of training? With the idea of gaining some insight into these questions we shall review some of the facts gained from case

* I wish to express my great appreciation of the aid of Professor L. L. Thurstone whose generosity made this study possible. Grateful acknowledgment is made of the aid of Mr. Ledyard Tucker in the use of the I.B.M. machines for obtaining the intercorrelations and the centroid factor loadings, and to both him and Mr. Harold Bechtoldt for aid in the testing of subjects.

histories of arithmetical prodigies and from the functional analysis of cases of acalculia or arithmetic disability.

Prior to the isolation of number ability by multiple factorial methods, there have been indications that this ability represents a functional unity in the mind. Scripture* in his article on arithmetical prodigies gave a brief account of more than fifteen individuals, dating from about 100 A.D., who gave evidence of phenomenal reckoning powers. Mitchell† also summarized what was known about mathematical prodigies and gives a brief account of his own case. Other psychologists have contributed to the knowledge of reckoning ability by psychological studies of individual cases.‡ Intensive experimental investigations of individual cases are lacking, most of the literature being biographical in nature. However, certain things which are of interest here seem to be clear. Mitchell states, "Skill in mental calculations is . . . . independent of general education; the mathematical prodigy may be illiterate or even densely stupid, or he may be an all-around prodigy and veritable genius."§ With the possible exception of memory for numbers, case histories of rapid mental calculators indicate that these individuals do not necessarily show any unusual ability in any other line of endeavor. Their most common characteristic is an unusual interest in and liking for numbers for their own sake. This interest is accompanied by a great deal of practice in their manipulation.

These data on mathematical prodigies seem to indicate that educational forces are of slight importance to this ability. A large proportion of the cases cited had little or no formal education and gave evidence of their unusual ability at a very early age. It might be argued that their interest and liking for numbers and numerical manipulation caused them to practice a great deal, and that this is essentially the counterpart of modern education. This may be true, but it seems more reasonable that an inherently superior native ability gave impetus to the interest, rather than vice versa. It is very unlikely that an average, or even below average, child could, by reason of interest, develop an ability to do numerical calculations which would transcend the ability of all his contemporaries.

* E. W. Scripture. Arithmetical prodigies. *American J. Psychol.*, 1891, 4, 1-59.

† Frank D. Mitchell. Mathematical prodigies. *Amer. J. Psychol.*, 1907, 18, 61-143.

‡ A. Binet. *Psychologie des Grands Calculateurs et Joueurs d'échecs*, Paris: Hachette, 1894, pp. 364 ff.

E. H. Lindley and A. L. Bryan. An arithmetical prodigy. *Psychol. Review*, 1900, 7, 135.

§ F. D. Mitchell, *op, cit.*, *p.* 131.

Studies of individuals with marked arithmetical disability are also very suggestive. Two cases have been reported in detail by Werner and Straus.* Case I revealed a specific deficiency in the finger schema and a general inability to grasp relations in space. Case II revealed a specific disturbance in the comprehension of visual patterns and in general a deficiency in grasping simultaneously the multiple aspects of a visual unity. All of these disturbances were found to differentiate a group of subjects selected for poor arithmetic ability from a group of subjects selected for good arithmetic ability. Guttmann, summarizing the evidence for congenital arithmetic disability arising from structural or functional anomalies of the brain, states, "There is little doubt that focal brain lesions may produce acalculia . . . . "†

The evidence from such cases of deficiency in number ability tends to indicate that the deficiency is more fundamental than mere lack of ability to manipulate numbers. This would be the case if the psychological process which we call number ability were inherent in the organism. Numbers, as a cultural invention, would be a much later development than a mental ability evolved by the slow processes of evolution.

It is reasonable, therefore, that tests like addition and multiplication really measure some more fundamental process than mere number manipulations *per se*. Hence it should be possible to devise tests of a non-numerical character which would measure this fundamental process. To do this one must make an hypothesis as to the nature of this process characterizing number ability.

If it is correct to assume that two senior students in high school know equally well that, say, eight and six are fourteen, and similarly for other possible number combinations, then the difference between their scores on a test of addition represents a difference in the rapidity with which the individuals can recall and manipulate these well-established associations. One might call this an agility in manipulating a symbolic system according to a specified set of rules, with two restrictions, that the symbolic system be familiar and that the rules be highly practiced. In a loose sense it might be descriptively referred to as mental agility, though this term would have more extensive implications than are desired here.

Heinz
* Henry Werner and Alfred Straus. Problems and methods of functional analysis in mentally deficient children. *J. Abn. & Soc. Psychol*, 1939, **34**, 37-62.

† E. Guttmann, Congenital arithmetic disability and acalculia, *British J. med. Psychol*, 1937, **16**, 18.

If number ability were basically such a process as this—the quick recollection and manipulation of well-established associations—then this fundamental process would transcend numerical operations. Inasmuch as "well-established associations" are products of educational and environmental forces, such forces would obviously play a role in the establishment of the ability. In view of the fact, however, that the learning curve has a horizontal asymptote, it is possible to set up tests of a non-numerical character in which the associations to be established are so simple that the asymptote is approached very early. These new tests would then measure primarily the rapidity with which these "well-established associations" could be manipulated.

From this hypothesis as to the nature of number ability certain deductions may be made which are susceptible to experimental verification. Briefly these may be stated as follows:

1) Tests involving new rules for manipulation of a symbolic system are better tests of number ability after practice than they are before.

2) Of several tests involving manipulation of a symbolic system, the one in which the symbolism is more familiar provides the better measure of number ability.

Another hypothesis as to the basic process involved in number ability may also be tested. It has been suggested by H. D. Landahl that the operation characterizing the tests defining the number factor is in the nature of a serial response, in that each response to a pair of numbers leads to the next response. This may be submitted to experimental verification by varying the length of the addition or multiplication problems. If the serial response hypothesis of number ability is correct, the addition of two digits would not be a measure of number ability, but the addition of four digits would.

Inasmuch as the tests of Identical Forms and Verbal Enumeration have not been very satisfactory as measures of perceptual speed, several cancellation tests were incorporated in the battery to see if they might not give a better definition of this factor. This clarification was of particular interest because of the fact that in a very simple number problem such as the addition of two digits, perception of the numbers and knowledge of the answers seem almost simultaneous. With good tests of the perceptual speed factor it is possible to determine what relation, if any, exists between it and number ability.

Tests were designed to check the validity of the hypotheses as to the nature of number ability and to determine what relation number ability bears to perceptual speed. These tests were incorporated with a variety of other tests covering the domain of known mental abilities, and the entire battery was analyzed by Thurstone's centroid method.

II. The Experiment

THE SUBJECTS

In the spring semester of 1939, the Chicago high schools offered the members of their senior classes a course in self-appraisal. Of their own volition the students in these classes took personality and aptitude tests with the intention of gaining some evaluation of their capabilities. Because the subjects for this experiment were taken from six such classes in six south-side high schools, it may be assumed that their motivation was fairly good and fairly constant. The six schools and the number of complete records obtained from each are as follows.

Englewood High School	87
Fenger High School	34
Hirsch High School	28
Hyde Park High School	21
Morgan Park High School	23
Parker High School	30
Total	223

Each class was visited for approximately an hour on each of five consecutive days, and from three to five tests were given to the subjects at each session.

TABLE 1

Tests for the Primaries

Number	Test	Primary
19	Identical Forms	P
20	Verbal Enumeration	P
21	Addition	N
22	Multiplication	N
23	Completion	V
24	Same-Opposite	V
25	Cards	S
26	Figures	S
27	Initials	M
28	Word-Number	M
29	Letter Grouping	I
30	Marks	I
31	Number Patterns	I
32	Arithmetic	D
33	Number Series	D
34	Mechanical Movements	D

THE TESTS

The tests used in this study may be divided into two groups: 1) the tests for the primaries and 2) the experimental tests.

The tests for the primaries, listed in Table 1, consisted of the sixteen tests in the American Council on Education *Tests for Primary Mental Abilities,* experimental edition, 1938. These tests were given by Miss Ruth Wright, of the Bureau of Child Study, Board of Education of Chicago, as part of the regular testing program of the high schools.

The experimental section of the battery consisted of eighteen tests selected and designed to answer specific problems. Each test is described below together with a statement of the purpose for which the test was designed.

Two-Digit Addition (1).—This test and the following two were designed to investigate the hypothesis that the number process was essentially in the nature of a serial response. If the hypothesis is correct, the saturations of tests (1), (2), and (3) on the number factor should progressively increase. The fore-exercise of this test, the whole of which consisted of 238 problems, was as follows:

Look at the following addition of problems.

8	9	3	8	5	3	7	14	18
6	7	5	6	2	2	2	4	1
—	—	—	—	—	—	—	—	—
14	16	9	14	7	5	7	18	19
		X				X		

Because they are wrong two of the answers are underlined with an X.

In the following problems cross out every answer that is wrong.

7	5	8	9	12	17	3	8	7	12	8
6	4	3	6	2	2	9	6	5	3	9
—	—	—	—	—	—	—	—	—	—	—
13	8	11	15	14	19	13	14	11	14	17

Three-Digit Addition (2).—This test consisted of 170 problems. The fore-exercise follows.

Look at the following addition problems.

8	5	4	9	7	6	4	8	5	9	4	2
7	3	8	3	2	5	1	6	9	3	6	6
6	1	4	2	6	2	7	3	2	1	7	4
—	—	—	—	—	—	—	—	—	—	—	—
21	9	17	14	15	14	13	17	16	13	16	12
		X			X	X				X	

Because they are wrong four of the answers are underlined with an X.
In the following problems cross out every answer that is wrong.

7	8	4	3	9	5	3	1	2	4	3	8
6	3	7	2	5	8	3	2	8	6	9	4
3	7	8	9	4	2	4	9	5	3	7	2
16	18	17	14	18	15	14	12	15	14	19	14

Four-Digit Addition (3).—This test consisted of 153 problems. The fore-exercise follows.

Look at the following addition problems.

5	6	7	8	4	2	9	2	3	5	7	7
6	3	1	5	1	3	8	9	5	6	3	8
6	3	7	8	1	3	2	6	5	3	8	5
4	7	2	1	8	3	9	2	8	5	3	7
21	18	17	22	15	11	27	19	21	18	21	27
	X			X		X			X		

Because they are wrong four of the answers are underlined with an X.
In the following problems cross out every answer that is wrong.

5	6	7	3	8	3	9	8	2	3	1	3
7	3	8	6	9	1	3	6	2	8	4	3
3	6	8	3	6	9	3	2	7	9	1	3
4	2	6	8	3	6	4	9	3	2	6	9
19	17	19	20	25	19	19	24	14	22	12	18

AB (4).—The *AB* and the *ABC* tests were designed to contrast with the Forms test to check the hypothesis that of several tests involving manipulation of a symbolic system, the one in which the symbolism is most familiar provides the best measure of number ability. The familiar alphabetical character of the elements in the two former tests contrasts markedly with the unfamiliar character of the elements used in the Forms test. Also the greater simplicity of the *AB* test compared to the *ABC* test would throw some light on the extent to which the speed element characterizes number ability. The fore-exercise of the *AB* test follows:

Only three letters, *A*, *B*, and *C*, are used in this test. They are combined in various ways, but there are only two rules. The rules are:

(1) A combination of any two different letters is equal to the third letter.

Examples: $AB=C$, $AC=B$, $BC=A$,
$BA=C$, $CA=B$, $CB=A$.

(2) If a letter is combined with itself, the combination is equal to that letter.

Examples: $AA=A$, $BB=B$, $CC=C$.

Below are some problems for you to practice on. If the answer to a problem is A, put a check (√) in the first column, if B put a check (√) in the second column, if C put a check (√) in the third column. The first three problems have been worked for you. Work them yourself to see that they are right and then go on with the others.

	A	B	C		A	B	C		A	B	C
AB			√	AA				CB			
CA		√		BA				AB			
BB		√		CA				AC			
AC				BC				BA			

This test consisted of 210 problems.

ABC (5).—This test consisted of 60 problems. The fore-exercise follows.

Only three letters, A, B, and C, are used in this test. They are combined in various ways, but there are only two rules. The rules are:

(1) A combination of any two different letters is equal to the third letter.

Examples: $AB=C$, $AC=B$, $BC=A$,
$BA=C$, $CA=B$, $CB=A$.

(2) If a letter is followed by itself, the combination is equal to that letter.

Examples: $AA=A$, $BB=B$, $CC=C$.

The example below has been worked out according to these rules.

$$ABA=B$$

Here is the way you solve the problem. Combine the first two letters. The combination $AB=C$. Then combine this C with the next letter, which is A. The combination $CA=B$.
Work the following examples and write the answers in the blanks.

$CBB=$ ——— $CAC=$ ———

You should have written C in the first blank and A in the second blank.

Here is a longer example which has been worked out.

$$CABC = A.$$

The first two letters, $C\,A$, equal B. This letter combined with the next gives $B\,B$, which is equal to B. This letter, combined with the next, gives $B\,C = A$.

Find the answer to the following example and write it in the blank.

$$BCAC = \text{——}$$

You should have written B.

Here are some more problems for you to practice on. Write the answers in the blanks.

$BCCAC = \text{—}$ $BABBA = \text{—}$ $ACBCA = \text{—}$

Forms (6).—This test was of exactly the same nature as the *ABC* test except that three non-meaningful geometrical designs replaced the letters *A, B,* and *C*.

Alphabet I (7).—This and the following two tests were designed to check the hypothesis that practice with a set of rules would improve a test as a measure of number ability. If the hypothesis is correct, the three tests, Alphabet I, Alphabet II, and Alphabet III should show progressively increasing validity as measures of number ability. The fore-exercise of Alphabet I follows.

In this test the letters of the alphabet are combined in various ways according to three rules.

(1) In a combination of two letters like *MP*, the letters are in the same order as they occur in the alphabet and there are several letters between them, *N* and *O*. Two letters like *RT* are also in alphabetical order and have one letter between them, *S*. Similarly one could write other combinations of two letters in alphabetical order in which there are three or more letters between them.

A combination of two letters in alphabetical order which has letters between them is equal to the letter in the alphabet which follows the second letter in the combination. For example, the two letters in the combination *MP* have letters between them, so the combination is equal to *Q* because *Q* follows *P* in the alphabet. Similarly, $RT = U$ because the letters *RT* have a letter between them and *U* follows *T* in the alphabet.

Other examples: $CE = F$, $HK = L$, $DV = W$.

(2) In a combination of two letters like *PM* the

letters are in backward order and have several letters between them, for example, *N* and *O*. Two letters like *TR* are also in backward order and have one letter between them, *S*. Similarly, one could write other combinations of two letters in backward order with three or more letters between them. A combination of two letters in backward order which has letters between them is equal to the letter in the alphabet which precedes the second letter of the combination. For example, the two letters *PM* have letters between them, and so the combination is equal to *L* because *L* precedes *M* in the alphabet. Similarly, $TR = Q$ because the letters *TR* have a letter between them and *Q* precedes *R* in the alphabet.

Other examples:

$$HC = B, \quad VN = M, \quad JH = G.$$

(3) A combination of two letters which have no letters between them is equal to the second letter of the pair.

$$\text{Examples: } OP = P, \quad GH = H, \quad RS = S,$$
$$PO = O, \quad HG = G, \quad SR = R.$$

Here are some problems for you to practice on. The first few have been answered for you. Check them to see that they are right and then go on with the others. Write the answer in the blank space.

$DF = G$	$JL = M$	$RS =$	$FH =$	$EC =$
$GJ = K$	$TS = S$	$LM =$	$BI =$	$YF =$
$RI = H$	$WY =$	$UT =$	$HJ =$	$OL =$

This test consisted of 192 problems.

Alphabet II (8).—Consisting of a second page of problems, this test was similar to Alphabet I. The subjects upon finishing Alphabet I had a rest period of several minutes and then took Alphabet II. The test consisted of 192 problems.

Alphabet III (9).—This test consisted of 192 problems similar to those in Alphabet I and Alphabet II. There was a brief period of rest between the tests Alphabet II and Alphabet III.

Digit Cancellation (10).—In conjunction with the next two tests, digit cancellation was designed to clarify the definition of a perceptual speed factor and thereby indicate what relation this factor would have to the number factor. The fore-exercise of this test was as follows:

Look at the rows of numbers below. A parenthesis has been put around each number 5.

3	7	2	(5)	9	0
7	(5)	2	8	6	3
1	0	8	6	2	7
4	6	9	(5)	(5)	0

Put a parenthesis around each number 5 in the following rows.

4	1	9	5	2	3
8	6	7	1	5	2
5	9	0	4	7	2
3	6	5	8	5	2

You are given several longer rows of numbers below. Put a parenthesis around each number 5.

8 7 6 0 3 5 2 1 0 5 8 4 7 9 3

Scattered X's (rowed) (11).—This test consists of ten pages of letters in rows through which *x*'s were scattered. The fore-exercise is below.

Look at the rows of letters below. A parenthesis has been put around each letter "*x*."

z	b	s	h	s	g
(x)	t	j	r	(x)	y
l	m	v	(x)	(x)	o
e	(x)	d	f	k	a

Put a parenthesis around each letter "*x*" in the following rows.

m	b	t	j	g	r
a	e	x	d	v	x
c	x	l	o	f	h
j	x	e	x	x	k

You are given several longer rows of letters below. Put a parenthesis around each letter "*x*."

v u t o c m z c f x m l z q k

Scattered X's (pied) (12).—This test consists of seven pages of pied letters with seven *x*'s on each page. The subjects were instructed to find as many *x*'s on a page as they were able and then to turn to the next page.

Identical Numbers (13).—This test, in conjunction with Highest Numbers and Digit Cancellation, was selected for the purpose of determining whether tests involving numbers and numerical concepts necessarily had a projection on number if they did not involve manipulation of the numbers. The test, the fore-exercise of which follows, consists of 45 columns with 29 three-digit numbers in each.

> The number at the top of the first column of figures is 634. A mark has been made under each 634 in the column. In the second column, a mark has been made under the 876, because 876 is the number at the top of that column. In the third column, the two 795's have been marked, because 795 is the number at the top of the third column.
> The number at the top of each of the other columns is repeated one or more times in that column. Find those numbers as quickly as possible and put a mark under each of them. Go right ahead.

634	876	795	423	279	374
693	643	583	837	363	282
850	328	795	115	643	663
634	932	189	423	279	539
513	879	342	528	375	314
398	375	795	969	470	375
696	470	896	274	887	576
634	697	247	423	699	374
574	876	319	627	291	850
628	294	468	423	983	677
634	982	543	962	585	846

Highest Number (14).—This test consists of 80 columns with 40 three-digit numbers in each. The fore-exercise is as follows:

> You are given five columns of numbers below. Find the highest number in each column and put a parenthesis around it. The first two columns have been done correctly. You mark the other three. Go right ahead. Do not wait for any signal.

1	2	3	4	5
298	189	229	189	149
142	237	340	376	580
389	(746)	187	234	689
527	642	246	427	327
462	248	546	167	486
127	543	827	349	682
482	329	628	256	595
(536)	735	821	453	337
227	670	342	113	428
162	167	119	297	321

Size Comparison (15).—It was thought desirable to set up a test of a "quantitative" nature but not involving numerical concepts to

see if number ability had a characteristic of quantitative thinking. The test consists of 69 items. The following is the fore-exercise.

Look at the following pairs of words.

sardine —(s)hark
(G)ermany —Cuba
(P)lanet —palace

A parenthesis has been put around the first letter of the word in each pair which means the larger of the two things. A shark is larger than a sardine; Germany is larger than Cuba; a planet is larger than a palace.
In the following pairs of words put a parenthesis around the first letter of that word in each pair which means the larger of the two things.

Texas —Maine	clock —watch	grape —lemon
thumb —foot	tree —forest	wrench —nut
inch —mile	rock —pebble	cup —barrel

Substitution I (16).—This test, in conjunction with Substitution II and Substitution III, was designed to see if an increasing familiarity with the translation of an arbitrary symbolism would have any

TABLE 2

Time Limit and Scoring Formulae used on the Experimental Tests

Number	Test	Time Limit in Minutes		Scoring Formula
		Fore-exercise*	Test	
1	Two Digit Addition - -	1	4	R-W
2	Three Digit Addition -	2	4	R-W
3	Four Digit Addition - -	2	4	R-W
4	AB - - - - - - - -	3	3	2R-W
5	ABC - - - - - - - -	8	5	2R-W
6	Forms - - - - - - -	13	5	2R-W
7	Alphabet I - - - - -	10	10	R
8	Alphabet II - - - - -		5	R
9	Alphabet III - - - -		5	R
10	Digit Cancellation - -	2	4	R
11	Scattered X's (rowed) -	1	4	R
12	Scattered X's (pied) -	2	4	R
13	Identical Numbers - -	2	3	R
14	Highest Number - - -	2	3	R
15	Size Comparison - - -	2	1.7	R-W
16	Substitution I - - - -	10	10	R
17	Substitution II - - -		5	R
18	Substitution III - - -		5	R

* The time limit for the fore-exercise was not rigidly adhered to except in the case of the substitution tests.

significance in relation to number ability. This test and the two following tests each consist of 90 words in code which are to be translated. The same code is used throughout and is given at the top of each test sheet.

Substitution II (17).—This test consists of a second page of code words similar to Substitution I and of the same length. The code is repeated on this test blank.

Substitution III (18).—This test is a third page of code words which are similar to those of the two preceding tests. The same code is also repeated on this test blank.

The time limit and scoring formula used for each of the eighteen tests comprising the experimental section of the battery are given in the following table.

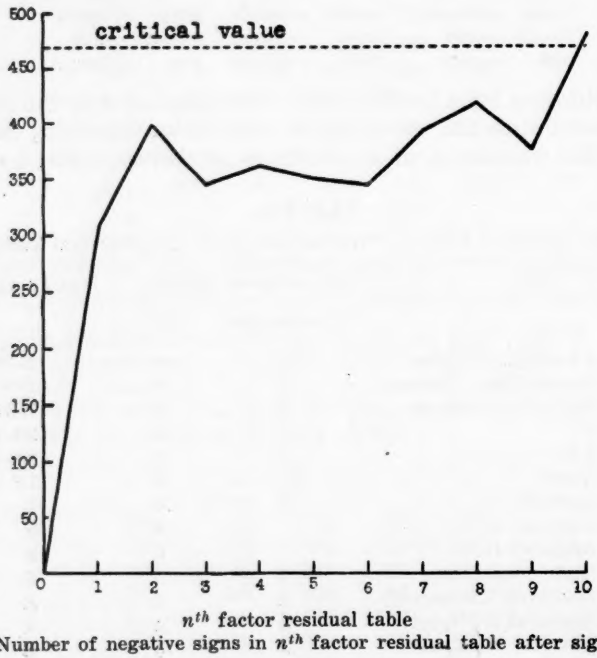

FIG. 1.—Number of negative signs in n^{th} factor residual table after sign change.

THE FACTOR ANALYSIS

In order to obtain the product moment correlation coefficients between tests by means of the punched card technique and I. B. M. machines, it was necessary that all test scores be expressed in no more than two digits. For this reason the scores of certain of the

TABLE 3
Correlations between the Tests

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
1																																		
2	849																																	
3	788	886																																
4	488	502	515																															
5	407	455	470	567																														
6	301	385	396	515	569																													
7	363	449	476	588	601	534																												
8	383	472	489	588	591	516	921																											
9	399	493	507	644	595	532	906	925																										
10	452	446	445	432	292	283	358	329	322																									
11	268	274	235	316	178	213	184	142	160	509																								
12	133	122	159	129	081	075	178	218	175	297	497																							
13	540	495	503	444	307	370	348	325	384	508	370	214																						
14	462	441	451	448	373	363	364	409	411	423	307	153	466																					
15	345	394	391	484	447	302	441	453	442	279	222	123	312	359																				
16	281	288	326	372	346	415	424	405	419	255	254	133	350	333	282																			
17	196	195	219	313	313	425	303	294	308	208	203	104	348	338	253	831																		
18	257	238	245	305	332	392	321	310	306	282	235	096	348	344	256	835	918																	
19	265	239	212	344	217	301	285	255	287	335	463	235	295	344	291	304	256	246																
20	354	347	310	404	316	336	318	303	304	358	298	176	455	281	536	192	165	179	300															
21	519	657	646	396	371	355	266	283	301	390	277	140	388	338	275	306	213	252	187	343														
22	733	703	639	408	358	284	345	355	337	384	192	141	405	356	213	235	167	251	221	328	556													
23	157	266	245	321	338	246	435	444	453	118	010	092	126	137	532	098	032	006	162	366	121	149												
24	259	334	323	440	458	277	484	527	511	181	001	–012	242	263	578	140	090	084	161	468	192	206	709											
25	175	253	225	345	351	307	442	413	407	218	226	123	134	259	266	259	149	163	476	220	210	187	236	222										
26	212	256	229	318	343	306	417	372	354	176	268	172	058	234	351	248	156	147	516	238	184	175	311	249	784									
27	155	223	178	170	132	171	266	274	256	110	077	085	167	124	167	200	097	147	119	126	150	208	163	190	240	145								
28	113	115	135	207	107	185	146	161	148	080	032	008	123	086	122	294	207	236	084	082	166	130	008	053	043	025	360							
29	226	327	292	391	305	317	422	433	423	327	311	208	259	268	178	267	171	166	309	215	296	230	218	295	264	283	192	067						
30	178	260	231	352	394	410	503	480	473	169	115	098	197	241	310	331	260	246	268	287	241	138	268	356	405	339	066	016	329					
31	256	306	325	246	152	260	225	226	241	188	242	200	194	330	151	188	121	134	347	123	354	272	029	052	335	305	108	031	301	384				
32	165	303	327	270	312	188	410	390	367	076	003	085	059	164	286	199	095	085	093	149	237	194	333	354	392	339	257	160	183	340	201			
33	263	361	358	320	329	218	480	472	467	166	089	151	143	224	355	174	070	058	167	203	274	251	383	384	402	343	168	096	258	398	252	595		
34	164	209	210	261	258	165	345	304	319	188	183	215	243	199	326	278	220	187	270	222	165	151	295	227	377	382	068	070	224	253	257	315	312	

tests were linearly translated to two digits, a process which does not affect the correlation coefficient. The correlation matrix is presented in Table 3.

The correlations were factored on the I.B.M. machines by Thurstone's centroid method. Application of a criterion developed for the purpose of detecting significant common factor variance in a matrix of intercorrelations indicated that ten factors were significant and the eleventh was not.* This criterion is based on the number

TABLE 4
Centroid Factorial Matrix

	I	II	III	IV	V	VI	VII	VIII	IX	X	h^2
1	64	–39	24	33	18	06	06	21	12	–12	84
2	72	–27	24	33	28	10	16	11	07	–09	90
3	71	–27	20	32	25	08	17	10	11	–04	84
4	72	03	–06	18	–07	–11	–06	06	–07	12	60
5	66	19	–13	17	05	06	–04	14	–13	15	58
6	62	03	–28	07	08	–09	–12	10	–24	14	58
7	77	37	–24	12	–05	–14	25	10	13	–05	92
8	77	36	–24	17	–06	–14	24	11	12	–12	93
9	78	34	–24	18	–07	–15	23	14	10	–08	93
10	54	–32	13	05	–23	–21	09	–04	04	12	54
11	42	–35	16	–26	–29	–32	04	–13	–11	03	61
12	29	–17	14	–24	–23	–20	23	–18	–04	–10	38
13	57	–35	02	19	–20	–07	02	–04	06	18	57
14	58	–19	03	03	–08	–09	–03	14	06	17	44
15	61	15	15	12	–26	23	–23	–05	04	08	62
16	59	–23	–52	–28	–03	25	–06	–05	13	–09	84
17	49	–31	–58	–28	–14	32	–15	10	09	06	92
18	51	–36	–57	–24	–10	30	–15	08	12	04	92
19	50	–10	13	–32	–13	–25	–21	12	–07	–09	53
20	53	–03	20	14	–26	08	–16	–12	–16	11	49
21	58	–30	14	12	28	08	08	–11	–19	09	61
22	57	–33	17	26	26	04	07	08	–03	–14	63
23	46	47	22	16	–26	25	–10	–10	–06	–22	71
24	54	43	13	29	–25	26	–13	–08	–12	–10	76
25	55	31	17	–43	20	–18	–19	16	14	07	77
26	54	29	23	–45	12	–13	–25	21	07	–13	79
27	32	05	–09	06	17	–16	–15	–35	19	–19	39
28	23	–06	–25	08	17	–06	–21	–37	17	–05	37
29	50	05	–02	–08	–04	–19	15	–08	–19	–10	37
30	53	25	–08	–20	06	06	12	07	–22	14	48
31	41	–08	13	–27	19	–12	15	07	–17	06	37
32	44	36	12	–10	28	22	11	–20	19	14	58
33	52	34	20	–11	18	16	24	–15	12	11	60
34	44	11	12	–26	–10	09	07	–06	11	03	33

* A paper presenting the logic upon which the criterion is based and the critical values for a given number of tests in the battery is in preparation.

of negative entries in the residual matrices after sign change. The critical value in the case of 34 variables is 468. When this value is attained no more significant common factor variance remains. The number of negative signs in the tenth factor residual table after sign change was 478. The number of negative signs in each residual table after sign change is presented in Figure 1.

The loadings on the ten centroid factors are presented in Table 4. This table was given to a disinterested person, who rearranged the table by rows and gave the variables a new set of numbers. The code remained in the possession of this person throughout the time the

TABLE 5
Rotated Factorial Matrix*

	N	*V*	*S*	*M*	*P*	*D*	*I*	*A*	*B*	*C*
1	74	02	05	–03	–01	–09	–06	–02	03	06
2	72	04	–03	02	–02	06	03	04	–02	02
3	66	01	–04	03	00	07	00	05	–01	07
4	10	09	09	06	09	–06	14	24	–04	30
5	06	11	04	–01	–14	04	23	27	01	25
6	02	–03	10	09	–05	–07	37	26	04	25
7	–02	–03	01	03	01	06	–03	64	02	03
8	02	00	00	04	00	–01	–04	66	02	–01
9	03	–01	00	01	00	–02	–02	66	01	04
10	18	–01	04	–01	46	–05	–05	04	–02	24
11	03	–01	19	–01	65	–03	06	–03	00	09
12	01	03	03	–03	55	07	–02	08	03	–13
13	22	04	–06	02	31	–09	–04	00	06	35
14	17	–04	16	–05	16	00	–01	04	06	29
15	–02	49	11	01	03	06	–08	–04	10	31
16	–01	00	02	15	02	04	00	03	69	–03
17	–08	00	03	00	–04	–03	01	–05	78	13
18	–01	–03	02	04	–03	–05	–01	–06	76	13
19	03	06	50	–03	28	–06	06	01	06	04
20	05	41	03	01	22	–02	10	–05	–04	30
21	45	04	–07	12	12	17	32	–10	–02	10
22	64	02	00	06	00	–05	12	00	00	–02
23	–06	66	03	–02	–04	03	–06	18	–02	–03
24	–05	64	–04	00	–09	–02	03	21	–04	12
25	–08	–08	65	04	–02	36	01	02	–04	06
26	–01	07	71	–02	–02	23	00	02	02	–08
27	03	01	08	53	05	–01	–05	04	–06	–04
28	–03	–03	–03	55	–02	–03	00	–06	05	08
29	04	04	05	04	28	02	22	30	–04	–08
30	–10	06	09	–11	03	29	30	24	08	04
31	17	–14	20	–09	20	25	26	04	–02	–06
32	–03	11	02	17	–11	57	–02	–02	–01	03
33	02	13	01	04	04	56	00	08	–04	–02
34	–06	15	15	–05	19	30	–09	00	15	00

* All entries have been multiplied by 100 to eliminate the decimal.

rotations were being made. The centroid matrix was rotated by means of extended vectors and the oblique rotational method. In rotating "blindly" in this manner the only criterion determining rotations was to go strictly to configuration. Seventeen rotations yielded a highly satisfactory simple structure in which no more rotations were evident. On June first the code was obtained and opened for the first time. No further rotations were made. The structure obtained by rotating "blindly" yielded the same psychologically meaningful factors as those obtained in numerous other studies in which the rotations were made with a knowledge of the names of the test vectors.

The factorial matrix obtained after rotation is presented in Table

TABLE 6

Direction Cosines of the Reference Vectors*

	N	*V*	*S*	*M*	*P*	*D*	*I*	*A*	*B*	*C*
I	21	16	17	09	16	12	11	20	13	15
II	–52	23	11	01	–33	25	00	44	–30	–05
III	35	33	28	–23	24	28	–15	–40	–46	–01
IV	36	15	–48	16	–29	–47	–04	24	–40	29
V	43	–45	11	36	–49	35	35	–15	–20	–20
VI	12	57	–34	–26	–41	32	–06	–39	62	–09
VII	16	–28	–58	–38	33	34	–03	46	00	–39
VIII	29	–26	39	–72	–44	–25	–08	24	14	02
IX	01	–25	14	22	–11	17	–90	–15	18	09
X	–34	–26	–11	–10	06	46	13	–26	–20	82

* All entries have been multiplied by 100 to eliminate the decimal.

TABLE 7

Cosines of the Angles Between the Reference Vectors*

	N	*V*	*S*	*M*	*P*	*D*	*I*	*A*	*B*	*C*
N	99									
V	–08	101								
S	–03	–11	100							
M	–12	–08	–02	101						
P	–17	01	–10	00	100					
D	–16	–07	–04	–04	07	102				
I	02	–01	–10	04	–02	01	100			
A	–12	–20	–19	–16	00	–31	11	99		
B	–02	15	–08	–22	–12	03	–21	–17	99	
C	–27	–04	04	08	–01	02	–02	–25	–24	99

* All entries have been multiplied by 100 to eliminate the decimal.

5, and the transformation matrix (Λ) leading from the centroid matrix (F_c) to the final rotated matrix (V) by the equation

$$V = F_c \Lambda$$

is given in Table 6.

The correlations between the reference vectors, obtained by the matrix multiplication ($\Lambda' \Lambda$), are presented in Table 7.

TABLE 8

Direction Cosines of the Primary Vectors*

	N	V	S	M	P	D	I	A	B	C
I	70	47	52	46	39	52	17	78	56	63
II	–39	37	16	02	–46	37	–12	40	–33	–12
III	28	36	20	–30	20	22	–22	–25	–58	–12
IV	34	21	–46	12	–28	–26	–19	16	–31	32
V	29	–33	06	32	–44	35	29	–07	–11	–17
VI	07	41	–30	–14	–38	25	02	–16	34	01
VII	15	–22	–46	–27	26	36	–13	26	–06	–20
VIII	15	–22	32	–61	–33	–15	–03	13	07	14
IX	10	–17	06	27	–09	20	–87	08	10	10
X	–13	–25	–17	–21	00	29	13	–10	–05	61

* All entries have been multiplied by 100 to eliminate the decimal.

The correlations between the primary vectors, obtained by the equation

$$R_{pq} = H H',$$

are shown in Table 9.

TABLE 9

Correlations Between the Primary Vectors*

	N	V	S	M	P	D	I	A	B	C
N	100									
V	24	100								
S	21	21	100							
M	28	19	17	100						
P	24	05	16	10	100					
D	33	24	20	22	03	100				
I	00	–02	10	00	06	–03	100			
A	44	35	34	36	14	46	–04	100		
B	27	03	22	32	21	15	21	37	100	
C	44	18	17	18	15	23	05	46	39	100

* All entries have been multiplied by 100 to eliminate the decimal.

The matrix H' is given by the equation

$$H' = (\Lambda' \Lambda)^{-1} D ,$$

where D is a diagonal matrix the entries in which are the normalizing constants of the corresponding columns of Λ. The columns of this matrix (H') contain the direction cosines of the primary vectors (see Table 8).

TABLE 10

Correlations of the Tests with the Primaries*

	N	*V*	*S*	*M*	*P*	*D*	*I*	*A*	*B*	*C*
1	91	23	23	22	22	21	–04	38	27	45
2	94	30	20	30	20	40	02	48	26	45
3	91	26	18	30	21	40	00	48	28	48
4	49	33	34	30	25	27	16	62	34	62
5	41	34	29	23	01	35	23	62	34	56
6	34	16	32	29	11	20	41	53	42	53
7	42	31	33	37	14	49	–06	95	38	46
8	44	34	32	38	13	44	–07	96	37	44
9	46	33	33	36	14	43	–04	96	38	48
10	47	14	21	15	59	13	–01	33	26	47
11	25	07	32	09	74	03	14	16	24	23
12	16	09	16	07	59	12	01	18	14	01
13	52	17	12	20	46	11	01	35	36	60
14	47	13	32	15	32	21	05	39	35	55
15	33	64	31	22	15	30	–03	43	28	53
16	28	08	25	45	22	22	17	41	91	34
17	18	01	21	26	16	08	21	29	96	44
18	24	–01	20	31	17	08	19	30	96	46
19	25	17	63	13	42	10	16	28	28	23
20	35	52	22	15	32	16	13	30	18	47
21	66	19	12	29	28	34	34	27	27	38
22	78	20	18	27	20	21	12	33	24	33
23	19	80	24	17	00	29	–09	47	04	17
24	26	80	19	21	–02	28	00	54	11	35
25	21	17	77	22	09	53	06	43	19	23
26	22	29	84	17	11	42	07	40	20	12
27	21	15	19	60	11	15	–06	26	14	08
28	12	04	05	59	04	06	02	15	24	16
29	28	21	27	22	36	23	22	46	21	17
30	19	23	31	10	11	47	33	50	31	29
31	32	01	33	05	29	35	29	25	18	13
32	25	31	18	34	–07	72	–05	39	12	20
33	33	36	21	24	08	74	–03	48	11	21
34	20	28	31	13	26	42	–04	83	26	19

* All entries have been multiplied by 100 to eliminate the decimal.

Applying the transformation (H') to the centroid matrix in the manner

$$R_{jp} = F_c H',$$

the correlations of the tests with the primary vectors are obtained (see Table 10). These correlations are the validity coefficients of the tests as measures of the primaries.

III. Interpretation and Discussion

The most interesting part of a factor analysis is the psychologi-

TABLE 11

Simplified Rotated Factorial Matrix*

	N	*V*	*S*	*M*	*P*	*D*	*I*	*A*	*B*	*C*
1	74									
2	72									
3	66									
22	64									
21	45						32			
23		66								
24		64						21		31
15		49								30
20		41			22					
26			71			23				
25			65			36				
19			50		28					
28				55						
27				53						
11			(19)		65					
12					55					
10					46					24
32						57				
33						56				
34					(19)	30				
6							37	26		25
30						29	30	24		
31			20		20	25	26			
8								66		
9								66		
7								64		
29					28		22	30		
5							23	27		25
17									78	
18									76	
16									69	
13	22				31					35
4								24		30
14										29

* All entries have been multiplied by 100 to eliminate the decimal.

cal interpretation of the factors obtained. To facilitate the interpretation we have reproduced the rotated matrix (Table 11), excluding all projections under .20 and rearranging the rows to make the structure more apparent.

The number factor.—Considering first the number factor, *N*, it may be seen that Two-Digit Addition has the highest projection, with Three-Digit Addition next, followed by Four-Digit Addition. The order of these three tests in their projection on number indicates that the simpler tests are better tests of this factor. This is borne out by

(1) Two-Digit Addition - - - -	.74	(22) Multiplication - - - - - - -	.64
(2) Three-Digit Addition - - -	.72	(21) Addition - - - - - - - - -	.45
(3) Four-Digit Addition - - - -	.66	(13) Identical Numbers - - - -	.22

the relation of Multiplication and Addition, the former being higher than the latter. The multiplication problems are the product of two digits by one digit, whereas the addition problems are the sum of six two-digit numbers. These results are strong evidence against the hypothesis that number ability is in the nature of a serial response process. The only other test having a projection greater than .20 is Identical Numbers, with a projection of .22. These results on the number factor indicate that it has a speed characteristic.

The rotated factorial matrix does not yield any information on the various hypotheses that the battery was designed to investigate because the primaries are oblique. However, information on these points may be obtained from the correlation matrix (Table 4) and from the validity coefficients of the tests (Table 10) as measures of the primary. These data are assembled in the following four tables.

The three alphabet tests were designed to check the hypothesis

TABLE 12

Correlations of Alphabet Tests with Number Tests

	Two-Digit Addition	Three-Digit Addition	Four-Digit Addition	Multiplication	Addition
(7) Alphabet I - - -	.363	.449	.476	.345	.266
(8) Alphabet II - -	.383	.472	.489	.355	.283
(9) Alphabet III - -	.399	.493	.507	.337	.301

that practice with a set of rules improves a test as a measure of number ability. If this is correct, the correlations of tests Alphabet I, Alphabet II, and Alphabet III with the number tests should progressively increase. This block of correlation coefficients is presented above in Table 12.

TABLE 13

Validity Coefficients of Alphabet Tests with the Number Primary

(7) Alphabet I	.42
(8) Alphabet II	.44
(9) Alphabet III	.46

The correlations all progressively increase with the one exception of the correlation between Alphabet III and Multiplication. Similarly, the validity coefficients of the three alphabet tests as measures of number ability show a slight increase as the subjects become more practiced with the set of rules. It is not to be expected that the amount of practice obtained by the subjects while taking these alphabet tests will yield a familiarity with the system in any way comparable with their familiarity with the number system.

The AB, ABC, and Forms tests were designed to throw light on the hypothesis that the more familiar or well-established the symbolism the better the test is as a measure of number. If this hypothesis is correct, the correlations of the AB and ABC tests with those for number ability should be distinctly higher than the correlation of the Forms test with the number tests. Table 14 shows that the predictions of the hypothesis are borne out without exception. The validity coefficients of these three tests as measures of number ability also support this conclusion.

TABLE 14

Correlations of AB, ABC, and Forms Tests with Addition and Multiplication Tests

	Two-Digit Addition	Three-Digit Addition	Four-Digit Addition	Multiplication	Addition
(4) AB	.488	.502	.515	.408	.396
(5) ABC	.407	.455	.470	.358	.371
(6) Forms	.301	.385	.396	.284	.355

TABLE 15

Validity Coefficients of AB, ABC, and Forms Tests with the Number Primary

(4) AB	.49
(5) ABC	.41
(6) Forms	.34

The consistent difference between the AB and the ABC test in the foregoing data also substantiate the conclusion that the simpler or less involved the manipulations called for in the test, the better is the test as a measure of number ability.

It is almost impossible to prove conclusively the original hypo-

thesis that number ability represents the rapidity with which well-established associations may be recalled and manipulated. The only conclusion which may safely be drawn is that the hypothesis is not disproved. All the data bearing on deductions from the hypotheses are in agreement with it.

The verbal factor.—The second factor, V, has the following four tests high on it:

(23)	Completion - - - - - - - -	.66
(24)	Same-Opposite - - - - - - -	.64
(15)	Size Comparison - - - - - -	.49
(20)	Verbal Enumeration - - - -	.41

There are no other tests with projections higher than .20. This factor is obviously the verbal factor. Completion and Same-Opposite are the two tests designed to isolate the verbal factor. Verbal Enumeration has been found previously to have a projection on the verbal factor. The fact that the new test, Size Comparison, has a projection on this factor is in agreement with the interpretation of the factor as verbal ability.

The space factor.—The third factor, *S*, is defined by the following tests:

(26)	Figures - - - - - - - - - -	.71
(25)	Cards - - - - - - - - - - -	.65
(19)	Identical Forms - - - - - -	.50
(31)	Number Patterns - - - - -	.20

This factor is readily seen to be the space factor, inasmuch as Figures and Cards are the two tests for this primary. Identical Forms has also been found previously to have a projection on the space factor. It is interesting to observe that none of the three tests are pure, the first two having projections on deduction and the third having a projection on perceptual speed. This indicates that we do not yet have clear insight into the psychological nature of this factor, or perhaps that the freedom given the subjects in adopting a work method results in some subjects using deductive ability to do the tasks in the Figures and Cards tests.

Memory.—The fourth factor, *M*, is obviously the Memory factor with no other tests high than the two included for the purpose of isolating this ability. Although this factor is called memory ability, it

(28)	Word-Number - - - - - - -	.55
(27)	Initials - - - - - - - - - -	.53

should be more strictly defined as rote learning, as both Word-Number and Initials are tests of rote learning with immediate recall. The

score is more representative of a point on the ascending learning curve than of a point on the descending forgetting curve.

Perceptual Speed.—The fifth factor, *P*, we have identified as perceptual speed. The tests with highest projections on this factor are the three designed for the purpose of clarifying the definition of perceptual speed.

(11) Scattered *x*'s (rowed) - -	.65	(19) Identical Forms - - - - -	.28
(12) Scattered *x*'s (pied) - - -	.55	(29) Letter Grouping - - - - -	.28
(10) Digit Cancellation - - - -	.46	(20) Verbal Enumeration - - -	.22
(13) Identical Numbers - - - -	.31	(31) Number Patterns - - - - -	.20

Certain suggestions are contained in these results which would be worth further investigation. There were considerably more cancellations per unit time in the Digit Cancellation test than in the Scattered *X*'s (rowed) test, so the indications are that the more scanning and the less cancelling called for by a test the better it is as a test of perceptual speed. The Scattered *X*'s (pied) test would probably be a better test of perceptual speed if administered in another manner. The subjects were instructed that there were only seven *x*'s on a page and that they might go on to the next page whenever they wanted to. It would probably improve the test if they were required to get *all* the *x*'s on each page before going on to the next, or perhaps, even better, if they were given a time limit of about twenty seconds on each page. An inspection of the fifth column of Table 11 indicates that all the tests with projections of .20 or greater involve the rapid scanning of the test content.

The Deductive Factor.—The sixth factor, *D*, may be identified as the deductive factor. The three tests included in the battery for the identification of this factor (Arithmetic, Number Series, Mechanical Movements) have their highest projections on it.

(32) Arithmetic - - - - - - - -	.57	(30) Marks - - - - - - - - - -	.29
(33) Number Series - - - - - -	.56	(31) Number Patterns - - - -	.25
(25) Cards - - - - - - - - - -	.36	(26) Figures - - - - - - - - -	.23
(34) Mechanical Movements -	.30		

Two of the space tests have significant projections on this factor for a possible reason already indicated. Two other tests, Marks and Number Patterns, have projections of .29 and .25 respectively on this factor. This is in agreement with the interpretation of this factor as deduction. It is interesting to note that with the definition of number ability by new tests with a greater speed characteristic than those in former studies, Arithmetic and Number Series have lower projections on the number primary.*

* Dr. H. O. Gulliksen has pointed out that these deductive tests seem to have a serial response character.

The inductive factor.—Those tests which have their highest projection on the seventh factor, *I*, are:

(6) Forms	.37	(31) Number Patterns	.26
(30) Marks	.30	(5) ABC	.23
(21) Addition	.32	(29) Letter Grouping	.22

Addition, which is primarily a number test, has a projection of .32 on this factor. Letter grouping and *ABC* also have projections greater than .20. Because of the small amount of variance accounted for by this factor, it is both difficult and risky to interpret. Inasmuch as Marks, Number Patterns, and Letter Grouping were included in the battery to identify the inductive plane, this factor might very tentatively be called induction.

Other factors.—The eighth factor, *A*, is primarily identified by the triplet of alphabet tests, a number of other tests having projections between .20 and .30. A factor defined by a triplet of tests so similar in nature as the three alphabet tests cannot be interpreted with any certainty. For that reason no attempt will be made here to indicate the psychological significance of this factor. Similarly, the ninth factor, *B*, is identified by the three substitution tests, with no other test having a projection on it as high as .20. This factor is another defined primarily by a triplet, and no attempt will be made to identify it.

The last factor, *C*, has very low variance and may be considered to be a residual factor.

Discussion

The reasons why such tests as *AB*, *ABC*, Forms, and the three alphabet tests have no projection on the number factor is that most of their variance is explained by other common factors, in this case primarily factors *A* and *C*, which have no "pure" tests to identify them. The result is that by rotating strictly to configuration, the number plane is passed through these tests, causing the configuration to be oblique. Hence, as may be seen from Table 9, the number primary correlated .44 with each of these two primaries, *A* and *C*. It is these correlations which have absorbed the potential projections of these tests on the number factor. It is assumed in this argument that the tests for the number factor do not call upon any non-number factor present in the alphabet tests. An alternative assumption is possible. It may be assumed that the alphabet tests are "pure" tests involving no number ability, but that the number tests are complex in that they involve not only the number factor but also the common factor defined by the alphabet tests. However, it seems more reasonable that

the simple number tests involve only one of the two factors and the more complex alphabet tests involve both of them.

Indications that word tests have significance to the number factor are apparent in several studies already made. One of the tests described by Thurstone in his monograph, *Primary Mental Abilities,** is Free Writing, the score on which is the total word count of a theme the subject is asked to write. This test had a projection of .295 on the number factor. In Thurstone's recent monograph, *Factorial Studies of Intelligence,*† the four highest tests on the word factor are Prefixes, First Letters, First and Last Letters, and Suffixes.

In the test Prefixes, the subjects were instructed to write all the words they could that began with "con." In First Letters the subjects were instructed to write all the words they could that began with the letter "s." In First and Last Letters the subjects were instructed to write all the words they could that began with "t" and ended with "e." In Suffixes they were instruction to write all the words they could that ended with "tion."

None of these tests had a projection on the number factor because in rotating strictly to configuration the number plane was passed through these tests. But, as in the case of the present study, the result was oblique primaries and the correlation between the axes absorbed the potential projections of these tests on number. The correlation between the number and word primaries in this battery was .33.

In another study, as yet unpublished, the three tests for the word factor were First Letters, Four-letter Words, and Suffixes. The number factor and word factor were correlated .47 in this study.

By refinement of these word tests it should be possible to devise word tests which do not involve number ability and, as opposed to these, others in which the content is verbal but which do not involve the word factor. It is not certain whether the manipulatory character or the associative character of these word tests is the basis of their relation to number. A type of word test for the investigation of this problem may be suggested here. The subject could be given a list of words which were the first members of pairs of words commonly associated, such as cup-saucer, man-woman, knife-fork. He could then be required merely to write the first letter of the associated word. Such a test would involve a minimum of manipulatory processes. On the other hand, a test like Free Writing, described above, probably has more of a manipulatory than an associative character.

* L. L. Thurstone. Primary mental abilities, *Psychometric Monographs*, 1938, No. 1. Chicago: Univ. Chicago Press, pp. 121.

† L. L. Thurstone. Factorial Studies of Intelligence, *Psychometric Monographs*, 1941, No. 2. Chicago: Univ. Chicago Press, pp. 94.

With regard to the interpretation of the factors in general, it should be pointed out that the identification of one of the factors as perceptual speed does not imply that it is *the* speed ability. Certain aspects of the classical problem of speed versus power may be reinterpreted in the light of the results of multiple factor analysis. There does not appear to be a single speed ability, but a number of them. If a test is composed of items of equal difficulty and, though not necessarily, of items of a low level of difficulty, then the scores on that test represent the amount done in a given time and measure a speed ability. In this sense at least three of the factors obtained in this study, number, space, and perceptual speed, are speed abilities.

VI

Summary and Conclusions

A battery of thirty-four tests was given to 223 high school seniors in the city of Chicago. The battery included sixteen tests for the identification of seven of the primary mental abilities and eighteen experimental tests for the investigation of particular hypotheses. The study was undertaken primarily to investigate the hypothesis that number ability represents the agility with which an individual can manipulate a symbolic system according to a specified set of rules, with two restrictions: that the symbolic system be familiar, and that the set of rules be highly practiced. Certain deductions were made from this hypothesis and non-numerical tests were designed to check these deductions. A secondary problem was to obtain a clearer identification of the perceptual speed factor in order that its relation to number ability might be determined.

The ten factors of the centroid matrix were rotated to simple structure without knowledge of the identity of the tests. Seven of the ten factors were given a psychological interpretation, two were triplets and non-interpretable, and one was a residual factor. The seven primaries that were given psychological interpretation were number, verbal, space, memory, perceptual speed, deduction, and induction.

Conclusions.—1) The number factor is most clearly identified by very simple number tests.

2) The perceptual speed factor is most clearly identified by cancellation tests, and the more scanning that is involved and the less cancelling the better the test is as a measure of perceptual speed.

3) A test involving manipulation of a symbolic system is a better measure of number ability the more familiar the symbolism.

4) A test involving operations according to a set of rules becomes a better measure of number ability with practice.

5) The data are in disagreement with the hypothesis that number ability is in the nature of a serial response.

6) The results are in agreement with the hypothesis that number ability is characterized by a facility in manipulating a symbolic system according to a specified set of rules.

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THE EVALUATION OF DETERMINANTS

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The numerical evaluation of determinants with a modern computing machine is discussed. Various methods are presented and their relations to each other are indicated. The methods presented parallel those developed in the previous paper on "The Solution of Simultaneous Equations." Especially emphasized are the Abbreviated Doolittle and the Compact methods. Additional topics include the evaluation of partially symmetric determinants by means of symmetric methods and the evaluation of determinantal ratios.

Introduction

Determinants are very useful in discovering the theoretical properties of the solutions of simultaneous equations, but they have not been found very useful in obtaining the numerical solutions. This is particularly true in least squares and correlation theory where approximate solutions only are demanded and where one usually has access to modern computing machines. Thus authors of text books on statistics frequently recommend non-determinantal methods for the numerical solution of normal equations. See, for example, (1, p. 67) (2, p. 36) (3, pp. 119-124). In a pervious article (4) the writer has indicated a number of these solutions.

It is possible to apply these methods to the evaluation of determinants. It is the purpose of this paper to show how this can be done. The reader who is familiar with the earlier paper should have little trouble in understanding the development even though the present paper is somewhat more condensed than the earlier one.

For purposes of brevity we use the fourth-order determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} & a_{43} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{vmatrix}, \qquad (1)$$

with the implication that the methods outlined are to be extended to the n'th order. As an illustration we take the determinant

$$\Delta = \begin{vmatrix} 1.0 & .4 & .5 & .6 \\ .4 & 1.0 & .3 & .4 \\ .5 & .3 & 1.0 & .2 \\ 6. & .4 & .2 & 1.0 \end{vmatrix}, \tag{2}$$

in which the elements are exact. It is evident from the definition of a determinant that the exact value of Δ is expressible in terms of four decimal places. This value is $\Delta = .3660$. In working with the different methods we attempt to carry the approximations to at least four places.

The determinant above is symmetric. However, it can be used to illustrate a non-symmetric determinant by assuming $a_{ij} \neq a_{ji}$. The reader will note that this determinant (2) is the determinant of the coefficients of the illustration of the paper on the solution of equations (4).

In all the methods which follow it is our plan to eliminate the first column, the second column, etc., in order. It is agreed that such an order is arbitrary. However, the technique is sufficiently general since if the rows and columns have the same order, they may be interchanged in all possible ways.

First Method. Method of Division

In this method, the elements of the first row of Δ are divided by a_{11}, the elements of the second row by a_{21}, etc. The first row of the resulting determinant is then multiplied by -1 and added to the second row, to the third row, and to the fourth row, in turn. We have then

$$\Delta = a_{11}\,a_{12}\,a_{13}\,a_{14} \begin{vmatrix} a'_{22\cdot1} & a'_{32\cdot1} & a'_{42\cdot1} \\ a'_{23\cdot1} & a'_{33\cdot1} & a'_{43\cdot1} \\ a'_{24\cdot1} & a'_{34\cdot1} & a'_{44\cdot1} \end{vmatrix}, \tag{3}$$

where $a'_{ij\cdot1} = \dfrac{a_{ij}}{a_{1j}} - \dfrac{a_{i1}}{a_{11}}$. If we treat the resulting determinant similarly, we get

$$\Delta = a_{11}\,a_{12}\,a_{13}\,a_{14}\,a'_{22\cdot1}\,a'_{23\cdot1}\,a'_{24\cdot1} \begin{vmatrix} a'_{33\cdot12} & a'_{43\cdot12} \\ a'_{34\cdot12} & a'_{44\cdot12} \end{vmatrix}, \tag{4}$$

where

$$a'_{ij\cdot12} = \frac{a'_{ij\cdot1}}{a'_{2j\cdot1}} - \frac{a'_{i2\cdot1}}{a'_{22\cdot1}};$$

and finally

$$\Delta = a_{11}\, a_{12}\, a_{13}\, a_{14}\, a'_{22\cdot 1}\, a'_{23\cdot 1}\, a'_{24\cdot 1}\, a'_{33\cdot 12}\, a'_{34\cdot 12}\, a'_{44\cdot 123} \tag{5}$$

is obtained by multiplying the entries in (5) or, if one prefers, by evaluating the determinant of (4) and multiplying by the indicated values.

The method is illustrated in Table 1 where

$$\Delta = (1.0)\,(.4)\,(.5)\,(.6)\,(2.1000)\,(.2000)\,(.2667)\,\{(7.3810)\,(3.8091) - (-.7440)\,(-1.1905)\} = .3660\,.$$

TABLE 1

METHOD OF DIVISION

1.0	.4	.5	.6
.4	1.0	.3	.4
.5	.3	1.0	.2
.6	.4	.2	1.0
1.0000	.4000	.5000	.6000
1.0000	2.5000	.7500	1.0000
1.0000	.6000	2.0000	.4000
1.0000	.6667	.3333	1.6667
	2.1000	.2500	.4000
	.2000	1.5000	−.2000
	.2667	−.1667	1.0667
	1.0000	.1190	.1905
	1.0000	7.5000	−.1000
	1.0000	−.6250	3.9996
		7.3810	−1.1905
		−.7440	3.8091
		1.0000	−.1613
		1.0000	−5.1198
			−4.9585
			.3660

This method is the least satisfactory of the various methods presented. The symmetry of the original determinant is lost with the first set of divisions. A large number of divisions is necessary while the method demands $n(n+1)$ rows (though but $n(n+1) - 3$ rows are used if the second-order determinant is evaluated directly).

Second Method. Method of Single Division

In the method of single division, the reduction is accomplished by a division of the first row by its leading element. Thus

$$\Delta = a_{11} \begin{vmatrix} 1 & b_{21} & b_{31} & b_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} & a_{43} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{vmatrix}, \qquad (6)$$

where $b_{i1} = \dfrac{a_{i1}}{a_{11}}$. We multiply the first row by $-a_{12}$ and add to the second, by $-a_{13}$ and add to the third, by $-a_{14}$ and add to the fourth and get.

$$\Delta = a_{11} \begin{vmatrix} a_{22\cdot1} & a_{32\cdot1} & a_{42\cdot1} \\ a_{23\cdot1} & a_{33\cdot1} & a_{43\cdot1} \\ a_{24\cdot1} & a_{34\cdot1} & a_{44\cdot1} \end{vmatrix}, \qquad (7)$$

where $a_{ij\cdot1} = a_{ij} - \dfrac{a_{i1}\, a_{1j}}{a_{11}}$. Similarly, we eliminate the first row of (7) and get

$$\Delta = a_{11}\, a_{22\cdot1} \begin{vmatrix} a_{33\cdot12} & a_{43\cdot12} \\ a_{34\cdot12} & a_{44\cdot12} \end{vmatrix}, \qquad (8)$$

where $a_{ij\cdot12} = a_{ij\cdot1} - \dfrac{a_{i2\cdot1}\, a_{2j\cdot1}}{a_{22\cdot1}}$. We thus get

$$\Delta = a_{11}\, a_{22\cdot1}\, a_{33\cdot12}\, a_{44\cdot123}\,. \qquad (9)$$

The method is illustrated in Table 2(a). To the first four rows of Δ are added the division row. The values in the next grouping are then obtained by subtracting from the element the product of the row heading and the columnar base. The division is made and entered, the next group of rows computed, etc.

$$\Delta = (1.0)\,(.84)\,(.7381)\,(.5903) = .3660\,.$$

A somewhat shorter variation utilizes the conventional method of evaluating a second order determinant. Thus

$$\Delta = (1.0)\,(.84)\,[(.7381)\,(.6095) - (-.1190)^2] = .3660\,.$$

This method is an approximation method since divisions are introduced. It maintains the symmetry property since $a_{ij\cdot1} = a_{ji\cdot1}$. It demands $\dfrac{n(n+3)}{2}$ or $[(\dfrac{n(n+3)}{2} - 2)]$ rows. The method is, essen-

TABLE 2

METHOD OF SINGLE DIVISION

(a)				(b)				(c)			
1.0	.4	.5	.6	1.0	.4	.5	.6	1.0	.4	.5	.6
.4	1.0	.3	.4	—	1.0	.3	.4	.4	1.0	.3	.4
.5	.3	1.0	.2	—	—	1.0	.2	.5	.3	1.0	.2
.6	.4	.2	1.0	—	—	—	1.0	.6	.4	.2	1.0
1.0	.4	.5	.6	1.0	.4	.5	.6	1.0	.4	.5	.6
	.84	.10	.16		.84	.10	.16		.84	.10	.16
	.10	.75	−.10		—	.75	−.10		.10	—	—
	.16	−.10	.64		—	—	.64		.16	—	—
	1.0000	.1190	.1905		1.0000	.1190	.1905		1.0000	.1190	.1905
		.7381	−.1190			.7381	−.1190			.7381	−.1190
		−.1190	.6095			—	.6095			−.1190	—
		1.0000	−.1612			1.0000	−.1612			1.0000	−.1612
			.5903				.5903				.5903
			.3660				.3660				.3660

tially, that of Chiò (5). If a leading element is 0, some other order must be chosen.

Third Method. Method of Single Division-Symmetric

In case the determinant is symmetric, it is not necessary to record the values below the main diagonal. The proper entry is found by subtracting from a_{ij} the product of b_{i1} and a_{j1}. The b_{i1} term is at the bottom of the column while the a_{j1} term is obtained by moving to the left to the main diagonal and taking the columnar heading.

The illustration is given in Table 2(*b*).

Fourth Method. Abbreviated Method of Single Division

In the abbreviated method of single division, the first row and the first column only of each new grouping is computed. The proper entry is finally obtained by subtracting one product out of each grouping. Thus

$$a_{43\cdot 12} = (.2) - (.5)(.6) - (.10)(.1905) = -.1190\,.$$

See Table 2(*c*) for illustration.

Fifth Method. Abbreviated Method of Single Division-Symmetric

The columnar entries of the fourth method may be eliminated if

the determinant is symmetric, since the top rows give the multipliers. Thus

$$a_{43\cdot 12} = (.2) - (.6)(.5) - (.10)(.1905) = -.1190$$

$$a_{44\cdot 123} = 1.0 - (.6)^2 - (.16)(.1905) - (.1190)(-.1612) = .5903\ .$$

See Table 3 (*a*).

TABLE 3

ABBREVIATED DOOLITTLE METHOD

(*a*)				(*b*)			
1.0	.4	.5	.6	1.0	.4	.5	.6
—	1.0	.3	.4	—	1.0	.3	.4
—	—	1.0	.2	—	—	1.0	.2
—	—	—	1.0	—	—	—	1.0
1.0	.4	.5	.6	1.0	.4	.5	.6
	.84	.10	.16	1.0	.4	.5	.6
	1.0000	.1190	.1905		.84	.10	.16
		.7381	−.1190		1.0000	.1190	.1905
		1.0000	−.1612			.7381	−.1190
			.5903			1.0000	−.1612
			.3660				.5903
							.3660

Sixth Method. Abbreviated Doolittle Determinantal Method

This method is essentially the same as the last method. The first row is repeated at the first grouping so that the technique is simply to subtract the products of elements in paired rows from the first element. This technique is easily carried out once it is understood. The final evaluation of the determinant, it is remembered, is

$$\Delta = (1.0)(.84)(.7381)(.5903) = .3660.$$

The Abbreviated Doolittle method can be expanded to give the conventional Doolittle method. It appears that Horst (6) was first to evaluate determinants with the use of the Doolittle method. He was chiefly interested in the correlation determinant but his method can be applied more generally. He derived the basic formula for evaluating determinants by any of the variations of the method of single division.

The equivalent of the formula (9) has been used more recently by Reiersøl (7) in computing all the principal minors of a determinant. His notation differs somewhat from the present notation but his technique results from an application of the methods and formulas of the method of single division.

Seventh Method. The Method of Multiplication and Subtraction

It was noticed in method one that the work was somewhat abbreviated by the use of (4) rather than (5). This leads to the suggestion as to the evaluation of the whole determinant by $ab - cd$ methods. This can be done.

The elements of the 2nd, 3rd, and 4th rows of Δ are multiplied by a_{11}, and $\frac{1}{a^3_{11}}$ is placed outside the determinant to compensate. The first row is then multiplied by $-a_{12}$ and added to the second, by $-a_{13}$ and added to the third, by $-a_{14}$ and added to the fourth. In the resulting determinant we have

$$\Delta = \frac{1}{a^3_{11}} \begin{vmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ 0 & A_{22\cdot 1} & A_{32\cdot 1} & A_{42\cdot 1} \\ 0 & A_{23\cdot 1} & A_{33\cdot 1} & A_{43\cdot 1} \\ 0 & A_{24\cdot 1} & A_{34\cdot 1} & A_{44\cdot 1} \end{vmatrix}, \tag{10}$$

where $A_{ij\cdot 1} = a_{ij}\, a_{11} - a_{i1}\, a_{1j}$.
Hence

$$\Delta = \frac{1}{a^2_{11}} \begin{vmatrix} A_{22\cdot 1} & A_{32\cdot 1} & A_{42\cdot 1} \\ A_{23\cdot 1} & A_{33\cdot 1} & A_{43\cdot 1} \\ A_{24\cdot 1} & A_{34\cdot 1} & A_{44\cdot 1} \end{vmatrix}. \tag{11}$$

Continuing this process we get finally

$$\Delta = \left(\frac{1}{a^2_{11}}\right)\left(\frac{1}{A_{22\cdot 1}}\right) A_{44\cdot 123}\,. \tag{12}$$

The successive steps in the computation are shown in Table 4(a).

This method has certain advantages. No divisions are needed for the forward solution so that the computation is eased. Also the value $\Delta = \frac{1}{(1.0)^2}\,\frac{1}{(.84)}\,(.30744)$ is exact, and the solution $\Delta = .3660$ is exact. Of course, exact values cannot be obtained beyond machine capacity. The method needs $\frac{n(n+1)}{2} + 1$ rows.

TABLE 4

METHOD OF MULTIPLICATION AND SUBTRACTION

(a)				(b)				(c)			
1.0	.4	.5	.6	1.0	.4	.5	.6	1.0	.4	.5	.6
.4	1.0	.3	.4	—	1.0	.3	.4	.4	1.0	.3	.4
.5	.3	1.0	.2	—	—	1.0	.2	.5	.3	1.0	.2
.6	.4	.2	1.0	—	—	—	1.0	.6	.4	.2	1.0
	.84	.10	.16		.84	.10	.16		.84	.10	.16
	.10	.75	−.10		—	.75	−.10		.10	—	—
	.16	−.10	.64		—	—	.64		.16	—	—
		.6200	−.1000			.6200	−.1000			.6200	−.1000
		−.1000	.5120			—	.5120			−.1000	—
			.30744				.30744				.30744
			.3660				.3660				.3660

Eighth Method. Method of Multiplication and Subtraction-Symmetric

In case the determinant is symmetric, the entries below the main diagonal may be left blank and the top entries used for computation as in method three. The illustration is given in Table 4(*b*).

Ninth Method. Abbreviated Method of Multiplication and Subtraction

The method can be abbreviated, as in method four, by recording only the entries of the first column and the first row in each matrix. The illustration is presented in Table 4(*c*). For example, the value of $A_{44\cdot 123}$ is

$$A_{44\cdot 123} = \{[1.0000 - (.6000)^2]\,.8400 - (.1600)^2\}\,.6200 - (-.1000)^2 = .30744.$$

Tenth Method. Abbreviated Method of Multiplication and Subtraction-Symmetric

In case symmetry is present it is possible to eliminate many of the rows of the last method. As a matter of fact, but $2n$ rows are needed to indicate the solution. The statement of the problem requires n rows. The next row gives the entries $a_{2j\cdot 1}$ (or $a_{i2\cdot 1}$), the next $a_{3j\cdot 12}$ (or $a_{i3\cdot 12}$), etc, The computational work is shown in Table 5(*a*). For example

$$A_{43\cdot 12} = [(.2000)(1.0000) - (.5000)(.6000)]\,.8400 - (.1000)(.1600) = -.1000\,.$$

A somewhat better form, though it requires an additional row, is indicated in Table 5(*b*). This differs from Table 5(*a*) only in the repetition of the first row at the end of the first n rows. It is then necessary only to take an element a_{ij}, multiply it by the element at the left of the $(n+1)$'st row and subtract the product of the i'th and j'th columnar entries in this row, multiply by the leading entry of the next row, etc. Thus

$$a_{44\cdot123} = \{[(1.0000)(1.0000) - (.6000)^2].8400 - (.1600)^2\}.6200 - (-.1000)^2 = .30744,$$

and

$$\Delta = \frac{1}{(1.0000)^2}\frac{1}{.8400}(.30744) = .3660.$$

TABLE 5

COMPACT METHOD

(a)				(b)			
1.0000	.4000	.5000	.6000	1.0000	.4000	.5000	.6000
—	1.0000	.3000	.4000	—	1.0000	.3000	.4000
—	—	1.0000	.2000	—	—	1.0000	.2000
—	—	—	1.0000	—	—	—	1.0000
	.8400	.1000	.1600	1.0000	.4000	.5000	.6000
		.6200	–.1000		.8400	.1000	.1600
			.30744			.6200	–.1000
			.3660				.30744
							.3660

This method is recommended as a most compact method in obtaining numerical approximations to the values of determinants of high order.

In all these methods a check column can be used.

The Evaluation of Partially Symmetric Determinants by Symmetric Methods

We may define a partially symmetric determinant to be one in which there is a symmetric minor, of order two or more, of any element of the principal diagonal. If this minor is of order one less than the order of the determinant, we may call the determinant "almost symmetric." Thus the determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{21} & a_{j1} \\ a_{12} & a_{22} & a_{j2} \\ a_{13} & a_{23} & a_{j3} \end{vmatrix}$$

is "almost symmetric" if $a_{12} = a_{21}$ since the minor of a_{j3} is then symmetric.

We wish to use the symmetric methods in the evaluation of Δ. This is done by inserting columns, for computational use, to make the first n rows and column symmetric. Thus we might evaluate Δ by the use of

$$\begin{matrix} a_{11} & a_{21} & a_{31} & a_{j1} \\ a_{j2} & a_{22} & a_{32} & a_{j2}\,, \\ a_{13} & a_{23} & - & a_{j3} \end{matrix} \quad \text{with } a_{3i} = a_{i3}\,,$$

and the solution is indicated by

$$\begin{matrix} A_{22\cdot 1} & A_{32\cdot 1} & A_{j2\cdot 1} \\ & - & A_{j3\cdot 12}\,, \end{matrix}$$

with $\Delta = \dfrac{A_{j3\cdot 12}}{a^2_{11}\, A_{22\cdot 1}}$.

If symmetry had been lacking in two columns, it would have been necessary to introduce two computational columns, etc.

This device can be adapted to the Abbreviated Doolittle method similarly.

The Simultaneous Evaluation of Almost Symmetric Determinants

It is desired to evaluate a number of determinants which are alike aside from the last column. If the number of these determinants is small, it is advised to use the method above and to add additional columns. Thus the evaluation of

$$\Delta = \begin{vmatrix} 1.0 & .4 & .5 & a_{j1} \\ .4 & 1.0 & .3 & a_{j2} \\ .5 & .3 & 1.0 & a_{j3} \\ .6 & .4 & .2 & a_{j4} \end{vmatrix} \tag{13}$$

is outlined in Table 6 by the compact method for different values of a_{ji}.

TABLE 6

ALMOST SYMMETRIC DETERMINANTS

						check
1.0	.4	.5	.6	.2	.8	3.5
.4	1.0	.3	.4	.4	.6	3.1
.5	.3	1.0	.2	.6	.4	3.0
.6	.4	.2	1.0	.8	.2	3.2
1.0	.4	.5	.6	.2	.8	3.5
	.84	.10	.16	.32	.28	.170
		.6200	–.1000	.3880	–.0280	.8800
			.30744	.36120	–.17640	.49224
		$\Delta =$	.3660	.4300	–.2100	.5860

When $a_{j1} = .6$, $a_{j2} = .4$, $a_{j3} = .2$, $a_{j4} = 1.0$, then $\Delta = .3660$.

When $a_{j1} = .2$, $a_{j2} = .4$, $a_{j3} = .6$, $a_{j4} = .8$, then $\Delta = .4300$.

When $a_{j1} = .8$, $a_{j2} = .6$, $a_{j3} = .4$, $a_{j4} = .2$, then $\Delta = -.2100$.

From the check column, when $a_{j1} = 3.5$, $a_{j2} = 3.1$, $a_{j3} = 3.0$, $a_{j4} = 3.2$, then $\Delta = .5860$.

If large numbers of these determinants are desired, however, it is preferable to divide the last column into its a_{j1}, a_{j2}, a_{j3}, a_{j4}, components, leaving a column for each component. The specified values a_{ji} can be inserted in the result. The form is shown in Table 7 where

TABLE 7 (*a*)

SIMULTANEOUS DETERMINANTS—ABBREVIATED DOOLITTLE METHOD

				a_{j1}	a_{j2}	a_{j3}	a_{j4}	check
1.0	.4	.5	.6	1	0	0	0	3.5
—	1.0	.3	.4	0	1	0	0	3.1
—	—	1.0	.2	0	0	1	0	3.0
—	—	—	1.0	0	0	0	1	3.2
1.0	.4	.5	.6	1	0	0	0	3.5
1.0	.4	.5	.6	1	0	0	0	3.5
	.84	.10	.16	–.40	1.00	0	0	1.70
	1.0000	.1190	.1905	–.4762	1.1905	0	0	2.0238
		.7381	–.1190	–.4524	–.1190	1.0000	.0000	1.0476
		1.0000	–.1612	–.6129	–.1612	1.3548	.0000	1.4193
			.5903	–.5966	–.2097	.1612	1.0000	.9451
			.3660	–.3699	–.1300	.1000	.6200	.5860

the fourth column is inserted to make possible symmetriic methods. The Abbreviated Doolittle method is shown in Table 7(a), while the "compact" method is presented in Table 7(b).

TABLE 7 (b)

SIMULTANEOUS DETERMINANTS—COMPACT

				a_{j1}	a_{j2}	a_{j3}	a_{j4}	check
1.0	.4	.5	.6	1	0	0	0	3.5
—	1.0	.3	.4	0	1	0	0	3.1
—	—	1.0	.2	0	0	1	0	3.0
—	—	—	1.0	0	0	0	1	3.2
1.0	.4	.5	.6	1	0	0	0	3.5
	.84	.10	.16	–.40	1.00	0	0	1.70
		.6200	–.1000	–.38	–.10	.84	0	.8800
			.30744	–.3108	–.1092	.0840	.5208	.49224
			.3660	–.3700	–.1300	.1000	.6200	.5860

It follows that the value of Δ is

$$\Delta = -.3700 a_{j1} - .1300 a_{j2} + .1000 a_{j3} + .6200 a_{j4} .$$

The values of Δ, for the value of a_{ji} used in Table 6, and for other values of a_{ji}, are given in Table 8.

TABLE 8

	$i=1$	$i=2$	$i=3$	$i=4$	Δ
A_{j1}	–.3700	–.1300	.1000	.6200	–.2100
a_{ji}	.6	.4	.2	1.0	.3660
a_{ji}	.2	.4	.6	.8	.4300
a_{ji}	.8	.6	.4	.2	–.2100
a_{ji}	3.5	3.1	3.0	3.2	.5860
a_{ji}	.5	.3	1.0	.2	.0000
a_{ji}	.4	1.0	.3	.4	.0000
a_{ji}	1.0	.4	.5	.6	.0000
a_{ji}	1.0	1.0	1.0	1.0	.2200
			etc.		

In general $\Delta = \sum a_{ji} A_{ji}$, where the A_{ji} are determined. It is to be noted that the quantities A_{ji} are the co-factors of a_{ji} and hence that the method is a method of determining the co-factors of the elements of a given column.

The Evaluation of Determinantal Ratios

In the solution of equations it is the ratio of two determinants, rather than the value of a given determinant, which is desired. The methods outlined above are useful in solving for determinantal ratios and, indeed, the forward solutions of the ten methods of solving simultaneous equations are immediately obtained. It is hence improper to call these "non-determinantal solutions" since these are the solutions which improved determinantal methods indicate.

Consider the equations

$$a_{11}\,x_1 + a_{21}\,x_2 + a_{31}\,x_3 = a_{j1}$$

$$a_{12}\,x_1 + a_{22}\,x_2 + a_{32}\,x_3 = a_{j2}$$

$$a_{13}\,x_1 + a_{23}\,x_2 + a_{33}\,x_3 = a_{j3}\,.$$

If we solve for x_3, the denominator is the determinant of the coefficients while the numerator is the denominator with a_{3i} replaced by a_{ji}. It can be shown that the determinantal ratios become

(a) in the method of division: $x_3 = \dfrac{a'_{j3\cdot 12}}{a_{33\cdot 12}}$;

(b) in the methods of single division: $x_3 = \dfrac{a_{j3\cdot 12}}{a_{33\cdot 12}}$;

(c) in the methods of multiplication and subtraction: $x_3 = \dfrac{A_{j3\cdot 12}}{A_{33\cdot 12}}$;

and these agree with the results obtained in the study of the solution of equations.

It is also possible to evaluate determinatal ratios when the determinants are not of the same order. In multiple correlation theory, for example, it is desired to evaluate the ratio of a determinant to one of its principal minors. As an illustration we desire to find $\dfrac{\Delta}{\Delta_{44}}$.

If the method of single division, or any of its variations, is used, we have

$$\frac{\Delta}{\Delta_{44}} = \frac{a_{11}\,a_{22\cdot 1}\,a_{33\cdot 12}\,a_{44\cdot 123}}{a_{11}\,a_{22\cdot 1}\,a_{33\cdot 12}} = a_{44\cdot 123}\,.$$

If the method of multiplication and subtraction, or any of its variations, is used, we have

$$\frac{\Delta}{\Delta_{44}} = \frac{A_{44\cdot 123}}{a_{11}\,A_{22\cdot 1}\,A_{33\cdot 12}}$$

Thus, from Table 3,

$$\frac{\Delta}{\Delta_{44}} = .5903$$

while from Table 5,

$$\frac{\Delta}{\Delta_{44}} = \frac{.30744}{(1.0000)\,(.8400)\,(.6200)} = \frac{.30744}{.5208} = .59032+.$$

In general

$$\left\{\begin{aligned} \frac{\Delta}{\Delta_{KK}} &= a_{KK\cdot 12\cdots\overline{K-1}} \\ \frac{\Delta}{\Delta_{KK}} &= \frac{A_{KK\cdot 12\cdots\overline{K-1}}}{a_{11}\,A_{22\cdot 1}\,A_{33\cdot 12}\cdots A_{\overline{K-1}\,\overline{K-1}\cdot 12\cdots\overline{K-2}}}. \end{aligned}\right.$$

Conclusion

It is shown that the same methods can be used in evaluating determinants that are used in solving simultaneous equations. Especially to be recommended are the "Abbreviated Doolittle" and "Compact" methods if the determinant is symmetric or almost symmetric.

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A NOTE ON THE DISCOVERY OF A *G* FACTOR BY MEANS OF THURSTONE'S CENTROID METHOD OF ANALYSIS

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A fictitious factor matrix including 16 tests and 3 factors, one of which was a *g* factor, was prescribed. From it two typical factor problems, including errors of sampling, were derived. Students in training, without awareness of the factor patterns, arrived at essentially correct solutions by the use of Thurstone's centroid method with rotation of axes. Errors in the calculated factor matrix were very close in size to the sampling errors in the correlation coefficients. It is concluded that a *g* factor need not escape detection by Thurstone's procedures if the criteria of complete simple structure are not demanded.

It is sometimes asserted that by virtue of its very nature Thurstone's centroid method of factor analysis cannot discover a *g* factor even when such a factor exists in a battery of tests. Thurstone's complete criteria for a unique solution was originally embraced in what he called "simple structure."* Complete simple structure calls for at least one zero loading in every row of the factor matrix and at least as many zeros in every column as there are factors, when the columns of the matrix correspond to the respective common factors and the rows to the respective tests. Since each factor, according to these requirements, must have at least one zero loading, it would follow that either there can be no *g* factor or there can be no unique solution.

As a matter of practice, however, complete simple structure not always obtains and yet unique solutions of a kind are possible. As a general rule, the configuration of test vectors in the common-factor space is such that the reference axes (usually orthogonal) find natural positions projecting through distinct clusters of those test vectors. The practical goal then becomes one of maximizing the number of zero factor loadings. When dealing with tests of abilities there is almost invariably a positive manifold to aid as another criterion. When these two conditions are fulfilled, a maximal number of zero loadings and a positive manifold, the reference axes usually have psychological meaning.

Even with these more liberalized criteria of a unique solution,

* Thurstone, L. L. The vectors of mind. Chicago: University of Chicago Press, 1925, 156.

however, it might be thought that there is grave danger of inadvertently missing a *g* factor. The mental set for zero loadings in every column of the factorial matrix must be recognized as a definite bias which might distort the solution. In this note is reported some evidence that a *g* factor will not necessarily escape Thurstone's instruments of analysis. The brief study also brings out one or two other confirming sidelights with regard to the validity of Thurstone's procedures.

A fictitious factor matrix (see Table 1) was set up with 3 common factors and 16 tests. Factor I was a *g* factor and factors II and III were not. To make the problem more exacting, two low factor loadings of .2 and .3 and two others of .4 were introduced into column 1 of the matrix. Ten zero loadings were introduced into the other two columns, since this is about the number that should be expected had there been complete simple structure in the matrix.

From the factor matrix the corresponding correlation matrix was computed. In order to make the problem more true to life, sampling errors were introduced at random assuming a population of 200. This process of introducing errors was carried out two times, making two slightly different correlation matrices. The error increments ranged from −.13 to +.16 in one matrix and from −.14 to +.14 in the other.

The two factor problems were presented to a class of four graduate students who were gaining their first instruction in factor methods. Two students solved each problem. They were told nothing concerning the source of the data, the number of common factors, or the fact that there was a *g* factor. From all they had learned previously in the course they should have been skeptical of the reality of a *g* factor except under special conditions. They were told to extract the centroid factors, to apply various tests of when to cease extracting, and then to rotate the reference axes observing the usual practical criteria, a positive manifold and a maximal number of zeros. According to all the criteria of the correct number of factors, the rank of the correlation matrix was three.

The procedure followed in rotation of axes was first the graphic method, with unextended vectors in planes, then later with extended vectors on the surface of a sphere. In all cases the first method gave the most valid results. In all cases there was no apparent escape from one *g* factor, in spite of the fact that the students tried to introduce zero loadings into every column. The results of the best solution are listed in Table 1. It is of considerable interest to note that the discrepancies between the original and the computed loadings are rather small for this solution. The range (see Table 1) is from −.17 to +.12.

The standard deviation of the discrepancies is .074, which is to be compared with the standard deviation of the original error increments, which was .075. The magnitude of error in the rotated loadings is no greater than that of the original correlation coefficients from which they came! This result should be heartening to those who, for lack of a rational method of finding standard errors of factor loadings, are worried lest factor loadings are heavily contaminated with errors of sampling.

In his recent volume on factor theory, Thomson gives several approximation tests of the extensiveness of the most extensive factor in a battery of tests.* A simple form of the test is to compare the largest latent root, approximated by means of the ratio $\sum r_{ii}$ (n where $\sum r_{ii}$ is the sum of all values in the correlation matrix, with the value of 1.00 in each diagonal cell, and n is the number of test variables) with successive sums of the communalities in order of size. In one of the two factor problems, for example, the largest latent root was estimated as 12.222. Summing the communalities, we find that the sum exceeds 12.222 only after all 16 communalities are included, at which point the sum is 13,498. From this we conclude that a *g* factor is present.

This simple and somewhat special problem does not of course settle the question with which we started. One cannot by any means conclude from it that Thurstone's method will always find *g* factors when they are present. What it does show is that a *g* factor will not necessarily escape detection when his procedures are employed, and we believe that the problem is typical enough to enable us to predict that a *g* factor if present will usually be discovered, if we do not demand simple structure but adhere to the more practical criteria of a positive manifold and a maximal number of zeros when abilities are concerned.

* Thomson, G. H. The factorial analysis of human ability. New York: Houghton Mifflin Company, 1939, Ch. XVII.

TABLE 1

FACTOR MATRICES, ORIGINAL AND CALCULATED, WITH COMMUNALITIES

	Original Matrix				Calculated Matrix			
Test	k_1	k_2	k_3	h^2	k'_1	k'_2	k'_3	h'^2
1	.9	.2	.0	.85	.94	.03	.00	.88
2	.8	.0	.3	.73	.78	—.08	.31	.70
3	.2	.7	.6	.89	.25	.73	.57	.91
4	.6	.6	.0	.72	.66	.55	—.09	.74
5	.7	.0	.4	.65	.68	—.02	.46	.66
6	.4	.5	.7	.90	.46	.55	.67	.94
7	.6	.5	.0	.61	.70	.40	—.05	.64
8	.7	.6	.2	.89	.83	.52	.06	.96
9	.7	.0	.7	.98	.69	.01	.68	.93
10	.5	.2	.8	.93	.52	.17	.84	.99
11	.6	.4	.0	.52	.65	.34	—.10	.54
12	.5	.6	.3	.70	.58	.58	.20	.71
13	.5	.0	.2	.29	.52	—.10	.23	.34
14	.4	.8	.0	.80	.48	.73	—.06	.76
15	.4	.6	.6	.88	.52	.63	.50	.91
16	.3	.0	.9	.90	.38	.08	.82	.82

TABLE 2

CRITERIA OF SUFFICIENT EXTRACTION OF FACTORS

Centroid Factor	Range Loadings	σ_ρ	Maximum k^2	$\Sigma\|\rho\|$	Σk^2	$\theta \geqq .938$ (Tucker)
I	......	......	.83	142.36	9.08	.479
II	.988	.088	.27	23.26	1.85	.806
III	.931	.062	.24	19.02	1.47	.681
IV	.486	.033	.08	4.77	.38	.956
		($\sigma_{\bar{r}} = .050$)				